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Cartesian Closedness of the Category of Fibrewise Convergence Spaces

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ABSTRACT. In this paper, we obtain the internal function space structure in the category of the fibrewise convergence spaces by means of the final structure. Moreover, we investigate cartesian closedness of the category of fibrewise convergence spaces which contains the category of fibrewise topological spaces as a full subcategory.

1. Introduction

The category **Top** of topological spaces and continuous maps fails to have some desirable properties, e.g. the product of two quotient maps need not be a quotient map and there is in general no natural function space topology i.e. the category is not cartesian closed.

Because of this fact, which is inconvenient for investigations in algebraic topology (homotopy theory), functional analysis (duality theory) or topological algebra (quotients), the category **Top** has been substituted either by well-behaved subcategories or by more convenient supercategories such as the category **Conv** of convergence spaces and continuous maps [9].

In this paper, we obtain the internal function space structure in $\mathbf{Conv}_B$, by means of the final structure. Moreover we investigate cartesian closedness of the category $\mathbf{Conv}_B$ which contains the category $\mathbf{Top}_B$ as a full subcategory.

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For general categorical background we refer to J. Adamek, H. Herrlich and G. E. Strecker [1], for the fibrewise theory to I. M. James [4,5,6] and for the convergence space to E. Binz [2].

2. Preliminaries

For any set X, we denote by $F(X)$ the set of all filters on X, and by $P(F(X))$ the power set of $F(X)$.

DEFINITION ([2]). Let X be a set. A function $\Gamma : X \to P(F(X))$ is said to be a *convergence structure* if the following properties hold for any point $x \in X$:

1) for any $x \in X$, $\langle x \rangle \in \Gamma(x)$, where $\langle x \rangle$ is the ultrafilter on X generated by $\{x\}$;
2) if $\mathcal{F} \in \Gamma(x)$, $\mathcal{G} \in F(X)$ and $\mathcal{F} \subset \mathcal{G}$, then $\mathcal{G} \in \Gamma(x)$;
3) if $\mathcal{F}, \mathcal{G} \in \Gamma(x)$, then $\mathcal{F} \bigcap \mathcal{G} \in \Gamma(x)$.

The pair (X, Γ) is named a *convergence space*. The filters in $\Gamma(x)$ are said to be *convergent* to x. For convergence spaces (X, Γ) and (Y, Γ'), the map $f : (X, \Gamma) \to (Y, \Gamma')$ is said to be *continuous at* $x \in X$ if for any $\mathcal{F} \in \Gamma(x)$, $f(\mathcal{F}) \in \Gamma'(f(x))$. And f is said to be *continuous* if f is continuous at each point x of X.

The class of all convergence spaces and continuous maps forms a category, which will be denoted by **Conv**. The category of all sets and functions between them will be denoted by **Set**.

Given an object B of a category $\mathbf{C}$, the category $\mathbf{C}_B$ of objects over B is defined as follows. An *object over* B is a pair (X, p) consisting of an object X of $\mathbf{C}$ and a morphism $p : X \to B$ of $\mathbf{C}$, called the *projection*. If X, Y are objects over B with projections p, q then a morphism $f : X \to Y$ of $\mathbf{C}$ is a *morphism over* B if $q \circ f = p$. Composition in $\mathbf{C}_B$ is defined according to the composition in $\mathbf{C}$. The object $X \in \mathbf{Top}_B$ is called the fibrewise topological space over a base B. The object $X \in \mathbf{Conv}_B$ is called a fibrewise convergence space over a base B.

In the category $\mathbf{Set}_B$, let X, Y be sets over B with projections p, q, respectively. A *fibre product* $X \times_B Y$ is the subset of $X \times Y$ consisting of pairs (x, y) such that $p(x) = q(y)$, with the projection r given by $r(x, y) = p(x) = q(y)$. In fact, $X \times_B Y$ is a product of X and Y in the category $\mathbf{Set}_B$.

The category $\mathbf{Conv}_B$ has an initial and final structures over $\mathbf{Set}_B$ [8]. If B is a topological space then the category $\mathbf{Top}_B$ is a bireflective full subcategory of $\mathbf{Conv}_B$ [8].

DEFINITION ([9]). A category $\mathbf{C}$ is called *cartesian closed* if

1) for each pair (A, B) of $\mathbf{C}$–objects, there exists a product $A \times B$ in $\mathbf{C}$, and
2) for each $\mathbf{C}$–object X, the functor $X \times - : \mathbf{C} \to \mathbf{C}$ has a right adjoint functor. That is, for each $Y \in \mathbf{C}$ there exist some $\mathbf{C}$–object Y^X and some $\mathbf{C}$–morphism $e : X \times Y^X \to Y$ satisfying following property : for each $\mathbf{C}$–object Z and each $\mathbf{C}$–morphism $f : X \times Z \to Y$, there exist a unique $\mathbf{C}$–morphism $\overline{f} : Z \to Y^X$ such that $e \circ (1_X \times \overline{f}) = f$.

3. Function Space Structures in $\mathbf{Conv}_B$

Now, we obtain a function space structure of the fibrewise convergence space by means of the final structure.

Let $X, Y \in \mathbf{Conv}_B$ and $\mathrm{map}_B(X, Y) = \coprod_{b \in B} \mathrm{map}(X_b, Y_b)$ as sets, where $\mathrm{map}(X_b, Y_b) = \{f \mid f : X \to Y$ is a function such that $f(X_b) \subset Y_b$ for $b \in B\}$. Let $\{f_i\}_{i \in I}$ be the family of all functions such that $f_i : X \times_B X_i \to Y$ is a function over B. Since the category $\mathbf{Conv}_B$ has a final convergence structure over $\mathbf{Set}_B$, we can define a final convergence structure on $\mathrm{map}_B(X, Y)$. Let Γ be the final convergence structure on $\mathrm{map}_B(X,Y)$ with respect to the family $\{\overline{f_i} \mid \overline{f_i} : X_i \to \mathrm{map}_B(X,Y)$ is a function over B such that $\overline{f_i}(z)(x) = f_i(x,z)\}_{i\in I}$. Then we have the following result.

THEOREM 1. *The space* $(\mathrm{map}_B(X,Y),\Gamma)$ *is a fibrewise convergence space over* B.

PROOF. Define a projection $p : \mathrm{map}_B(X,Y) \to B$ by $p(g) = b$ for $g \in \mathrm{map}(X_b, Y_b)$. Then p is a well–defined function. Now we will show this projection p is continuous. Let $\overline{f}_i \in \{\overline{f_i} \mid \overline{f_i} : X_i \to \mathrm{map}_B(X,Y)$ is a function over B such that $\overline{f_i}(z)(x) = f_i(x,z)\}_{i\in I}$. Then $p\circ\overline{f_i} = p_i$ for all $i \in I$, where p_i is the projection from X_i to B. Since the family $\{\overline{f_i}\}_{i\in I}$ is a final family and p_i is continuous for all $i \in I$, the projection p is continuous. Therefore the space $(\mathrm{map}_B(X,Y),\Gamma)$ with the projection p is a fibrewise convergence space over B.

Since the category $\mathbf{Conv}_B$ has an initial structure, we can consider the subspace structures. Let $X, Y \in \mathbf{Conv}_B$. Then the fibre product $X \times_B Y$ with the initial structure with respect to the family $\{\mathrm{pr}_1 : X \times_B Y \to X, \mathrm{pr}_2 : X \times_B Y \to Y\}$, is the product of X and Y in the category $\mathbf{Conv}_B$ [7].

THEOREM 2. *If* $\{f_i : X_i \to Y\}_{i\in I}$ *is a final family in* $\mathbf{Conv}_B$ *then* $\{1 \times f_i : W \times_B X_i \to W \times_B Y\}_{i\in I}$ *is also a final family in* $\mathbf{Conv}_B$ *for any* $W \in \mathbf{Conv}_B$.

PROOF. Clearly the projections $\mathrm{pr}_1 : W \times_B X_i \to W$ are continuous for all $i \in I$. Since the map $f_i : X_i \to Y$ is continuous, $f_i \circ \mathrm{pr}_2 : W \times_B X_i \to Y$ is also continuous for all $i \in I$. But $\mathrm{pr}_2 \circ (1 \times f_i) = f_i \circ \mathrm{pr}_2 : W \times_B X_i \to Y$ and $\mathrm{pr}_1 \circ (1 \times f_i) = \mathrm{pr}_1 : W \times_B X_i \to W$. Hence the map $1 \times f_i : W \times_B X_i \to W \times_B Y$ is continuous for all $i \in I$.

Next, we investigate $1 \times f_i$ is a map over B. Let p_i and q be projections of X_i and Y, respectively. Then $q \circ f_i = p_i$ for all $i \in I$. Hence for the point $(w, x) \in W \times_B X_i$, we have $(q \circ \mathrm{pr}_2) \circ (1 \times f_i)(w, x) = (q \circ \mathrm{pr}_2)(w, f_i(x)) = q(f_i(x)) = (q \circ f_i)(x) = p_i(x) = (p_i \circ \mathrm{pr}_2)(w, x)$. Thus $(q \circ \mathrm{pr}_2) \circ (1 \times f_i) = p_i \circ \mathrm{pr}_2$ for all $i \in I$. That is, $1 \times f_i$ is a map over B.

It remains to show that for any $Z \in \mathbf{Conv}_B$ and any function $h : W \times_B Y \to Z$ in $\mathbf{Set}_B$, h is a map in $\mathbf{Conv}_B$ if and only if $h \circ (1 \times f_i) : W \times_B X_i \to Z$ is a map in $\mathbf{Conv}_B$ for all $i \in I$. The only if part is trivial. Suppose that $h \circ (1 \times f_i) : W \times_B X_i \to Z$ is a map in $\mathbf{Conv}_B$ for all $i \in I$. Since h is a map in $\mathbf{Set}_B$, it is enough to show that h is continuous. Let the filter generated by the filter base $\mathcal{F} \times_B \mathcal{G} = \{F \times_B G | F \in \mathcal{F} \text{ and } G \in \mathcal{G}\}$ converge to (w, y) in $W \times_B Y$. Then the filter $\mathcal{F}$ converges to w in W, and the filter $\mathcal{G}$ converges to y in Y.

Suppose $(w, y) \notin \bigcup_{i \in I} (1 \times f_i)(W \times_B X_i)$. Then the filter $\mathcal{F} \times_B \mathcal{G}$ contains the ultrafilter $\langle (w, y) \rangle$ generated by the singleton set $\{(w, y)\}$. Then $\mathcal{F} \times_B \mathcal{G} = \langle (w, y) \rangle$. Hence $h(\mathcal{F} \times_B \mathcal{G}) = h(\langle (w, y) \rangle) = \langle h(w, y) \rangle$. Thus the filter $h(\mathcal{F} \times_B \mathcal{G})$ converges to $h(w, y)$. Hence h is continuous.

Next, suppose that $(w, y) \in (1 \times f_{i0})(W \times_B X_{i0})$ for some $i_0 \in I$. Let $(1 \times f_{i0})(w, x_{i_0}) = (w, y)$ for some $x_{i_0} \in X_{i0}$. From the fact that $\mathcal{G}$ converges to y and Y has the final convergence structure, we know that $\mathcal{G} \supset \langle y \rangle$ or $\mathcal{G} \supset \bigcap_{i=1}^{n} f_{k_i}(\mathcal{H}_{k_i})$ where $\mathcal{H}_{k_i}$ is convergent to x_{k_i} in X_{k_i} and $f_{k_i}(x_{k_i}) = y$.

If the filter $\mathcal{G}$ contains the filter $\langle y \rangle = \langle f_{i0}(x_{i_0}) \rangle = f_{i0}(\langle x_{i_0} \rangle)$, then the filter $\mathcal{F} \times_B \mathcal{G}$ contains the filter $\mathcal{F} \times_B f_{i0}(\langle x_{i_0} \rangle) = (1 \times f_{i0})(\mathcal{F} \times_B \langle x_{i_0} \rangle)$. Thus $h(\mathcal{F} \times_B \mathcal{G}) \supset h \circ (1 \times f_{i0})(\mathcal{F} \times_B \langle x_{i_0} \rangle)$. By the hypothesis, $h \circ (1 \times f_{i0})$ is continuous. Since the filter $\mathcal{F} \times_B \langle x_{i_0} \rangle$ converges to (w, x_{i_0}), the filter $h \circ (1 \times f_{i0})(\mathcal{F} \times_B \langle x_{i_0} \rangle)$ converges to $h \circ (1 \times f_{i0})(w, x_{i_0})$.

Thus the filter $h(\mathcal{F} \times_B \mathcal{G})$ converges to $h \circ (1 \times f_{i0})(w, x_{i_0})$. But $h \circ (1 \times f_{i0})(w, x_{i_0}) = h(w, f_{i0}(x_{i_0})) = h(w, y)$. Hence h is continuous. If $\mathcal{G} \supset f_{k_1}(\mathcal{H}_{k_1}) \bigcap \cdots \bigcap f_{k_n}(\mathcal{H}_{k_n})$ for some filters $\mathcal{H}_{k_i}$ which converges to x_{k_i} in X_{k_i} and $f_{k_i}(x_{k_i}) = y$, then $\mathcal{F} \times_B \mathcal{G} \supset \bigcap_{i=1}^{n} \mathcal{F} \times_B f_{k_i}(\mathcal{H}_{k_i}) = \bigcap_{i=1}^{n} (1 \times f_{k_i})(\mathcal{F} \times_B \mathcal{H}_{k_i})$. Thus $h(\mathcal{F} \times_B \mathcal{G}) \supset h\left(\bigcap_{i=1}^{n} (1 \times f_{k_i})(\mathcal{F} \times_B \mathcal{H}_{k_i})\right) = \bigcap_{i=1}^{n} h \circ (1 \times f_{k_i})(\mathcal{F} \times_B \mathcal{H}_{k_i})$. Note that $h \circ (1 \times f_{k_i})$ is continuous. Since $\mathcal{F} \times_B \mathcal{H}_{k_i}$ converges to (w, x_{k_i}) for all $i = 1, \cdots, n$, the filter $h \circ (1 \times f_{k_i})(\mathcal{F} \times_B \mathcal{H}_{k_i})$ converges to $h \circ (1 \times f_{k_i})(w, x_{k_i}) = h(w, y)$, and hence the filter $\bigcap_{i=1}^{n} h \circ (1 \times f_{k_i})(\mathcal{F} \times_B \mathcal{H}_{k_i})$ converges to the point $h(w, y)$. Thus $h(\mathcal{F} \times_B \mathcal{G})$ converges to $h(w, y)$. Hence h is continuous. In all cases, we have that h is continuous. Hence the theorem follows.

4. Cartesian Closedness of the Category $\mathbf{Conv}_B$

In order to investigate the cartesian closedness of the category of fibrewise convergence spaces, we first show that the evaluation map $e : X \times_B \mathrm{map}_B(X, Y) \to Y$ is continuous.

THEOREM 3. *The map* $e : X \times_B \mathrm{map}_B(X, Y) \to Y$ *defined by* $e(x, f) = f(x)$ *is continuous.*

PROOF. Clearly e is a map in $\mathbf{Set}_B$. Let $f_i : X \times_B X_i \to Y$ be an arbitrary map in $\mathbf{Conv}_B$. Then, as a function, f_i belongs to the family $\{f_i\}_{i \in I}$ of all functions such that $f_i : X \times_B X_i \to Y$ is a function over B. Since $\{\overline{f}_i | \overline{f_i} : X_i \to \mathrm{map}_B(X, Y)$ with $\overline{f}_i(z)(x) = f_i(x, z)\}_{i \in I}$ is a final family, $\{1 \times \overline{f_i} : X \times_B X_i \to X \times_B \mathrm{map}_B(X, Y)\}_{i \in I}$ is also a final family by the above theorem. Since $e \circ (1 \times \overline{f_i}) = f_i : X \times_B X_i \to Y$ and f_i are continuous for all $i \in I$, the evaluation map e is continuous.

Note that the fibre product $X \times_B Y$ with the initial structure is the product of X and Y in the category $\mathbf{Conv}_B$. Now, we will show that the functor $X \times_B - : \mathbf{Conv}_B \to \mathbf{Conv}_B$ has a right adjoint functor.

THEOREM 4. *For any convergence space X over B, the functor $X \times_B - : \mathbf{Conv}_B \to \mathbf{Conv}_B$ has a right adjoint functor. That is, for each $Y \in \mathbf{Conv}_B$, there exist $Y^X \in \mathbf{Conv}_B$ and a map $e : X \times_B Y^X \to Y \in \mathbf{Conv}_B$ satisfying the following property : for any $Z \in \mathbf{Conv}_B$ and any map $f : X \times_B Z \to Y \in \mathbf{Conv}_B$ there is a unique $\overline{f} : Z \to Y^X \in \mathbf{Conv}_B$ with $e \circ (1_Z \times \overline{f}) = f$.*

PROOF. Take $Y^X = (\mathrm{map}_B(X,Y), \Gamma)$ and the map e as in the above theorem. For any $(Z, r) \in \mathbf{Conv}_B$ and any function $f : X \times_B Z \to Y \in \mathbf{Conv}_B$, define the function $\overline{f} : Z \to Y^X$ by $\overline{f}(z)(x) = f(x,z)$. Then for any $z \in Z_b$ and $x \in X_b$, $\overline{f}(z)(x) = f(x,z) \in Y_b$. Thus $\overline{f}$ is a function from X_b to Y_b. That is, $\overline{f}(z) \in \mathrm{map}(X_b, Y_b)$. Hence $(p \circ \overline{f})(z) = b = r(z)$, where p is the projection of the convergence space Y^X. Then $\overline{f} \in \mathbf{Set}_B$. Since $\overline{f}$ is continuous, $\overline{f} \in \mathbf{Conv}_B$. Clearly $e \circ (1 \times \overline{f}) = f$. It remains to show that such a map $\overline{f}$ is unique. Suppose that there exists another map $g : Z \to Y^X \in \mathbf{Conv}_B$ with $e \circ (1 \times g) = f$. Then $g(z)(x) = e(x, g(z)) = (e \circ (1 \times g))(x,z) = f(x,z) = \overline{f}(z)(x)$ for all $(x,z) \in X \times_B Z$. Hence $g(z) = \overline{f}(z)$ for all $z \in Z$. Thus $g = \overline{f}$. Hence such a map $\overline{f}$ is unique.

From the above theorem, we have the following result.

THEOREM 5. *The category $\mathbf{Conv}_B$ is cartesian closed.*

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