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A Numerical Method for the Minimum Norm Solution to the First Kind Integral Equations

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ABSTRACT. This paper introduces a numerical method approximating the minimum norm solution to the first kind integral equation $Kf = g$ with its kernel satisfying a certain property, where g belongs to the range space of K. Most of the existing expansion methods suffer from choosing a set of basis functions, whereas this method automatically provides an optimal set of basis functions approximating the minimum norm solution of $Kf = g$. Perturbation results and numerical experiments are also provided to analyze this method.

1. Introduction.

Unless otherwise stated, throughout this paper it is assumed that $K : L_2[a,b] \to L_2[c,d]$ is an integral operator defined by $(Kf)(x) = \int_a^b k(x,y)f(y)\,dy$ with the kernel function $k(x,y) \in L_2([c,d] \times [a,b])$, k_x is defined to be $k_x(y) = k(x,y)$, g belongs to the range space of K, and all functions are real-valued for simplicity of exposition. Then it is clear that K is a bounded linear compact operator and the least squares solution of minimum norm to $Kf = g$ is the same as its minimum norm solution. Here, an element $f_0 \in L_2[a,b]$ is said to be a *least squares solution of minimum norm*(LSSMN) to $Kf = g$ if f_0 is a least squares solution to $Kf = g$ and $\| f_0 \| \leq \| f_1 \|$ for any other least squares solution f_1 to $Kf = g$.

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Some notations and basic terminologies used in this paper are provided below. The symbol $(\cdot,\cdot)$ and $\|\cdot\|$ will denote the generic inner product and the associated norm in a Hilbert space, respectively. If M is a subset of a Hilbert space H, then $M^{\perp}$ will denote the *orthogonal complement* of M, i.e.,

$$M^{\perp} = \{f \in H \mid (f,g) = 0 \text{ for all } g \in M\}.$$

$\vee M$ will denote the *closed linear span* of M. If T is a bounded linear operator from a Hilbert space H_1 into a Hilbert space H_2, then T^* will denote the *adjoint* of T. That is, $T^* : H_2 \to H_1$ is the bounded linear operator defined by $(Tx,y) = (x,T^*y)$ for all $x \in H_1$ and $y \in H_2$. $N(T)$ and $R(T)$ will denote the *null* and *range* space of T, respectively. The operator $T^{\dagger} : R(T)+R(T)^{\perp} \to H_1$ called the *Moore-Penrose generalized inverse* of T assigns to each $g \in R(T) + R(T)^{\perp}$ the least squares solution of minimum norm(LSSMN) of the equation $Tf = g$ [3].

This paper considers a numerical method for finding the minimum L_2-norm solution to the first kind integral equation $(Kf)(x) = g(x)$ for all $x \in [c,d]$. Section 2 provides some theoretical results for a numerical method, Section 3 contains perturbation results for this method, and Sections 4 and 5 contain numerical results and conclusions, respectively.

2. Theoretical Results and Numerical Method.

It was assumed that the kernel $k(x,y)$ of the integral operator K is in $L_2([c,d] \times [a,b])$. Then, by Fubini's theorem, $\int_a^b |k(x,y)|^2\,dy = \int_a^b |k_x(y)|^2\,dy$ exists for almost all $x \in [c,d]$ and is a measurable function of x. But it is not true that $\int_a^b |k_x(y)|^2\,dy$ exists for all $x \in [c,d]$ or that $(Kf)(x) = \int_a^b k(x,y)f(y)\,dy$ exists for all $x \in [c,d]$. For this reason, we define a class of integral operators K with kernel $k(x,y)$

satisfying a special property, so that a numerical method approximating the minimum norm solution to $(Kf)(x) = g(x)$ for all $x \in [c, d]$ can be considered.

DEFINITION 2.1. The integral operator K with kernel function $k(x, y)$ is said to satisfy *Property (c)* if

(a) $k_x \in L_2[a, b]$ for all $x \in [c, d]$, and
(b) if $(Kf)(x) = 0$ a.e. for a $f \in L_2[a, b]$, then $(Kf)(x) = 0$ for all $x \in [c, d]$.

From the condition (a) and Schwarz inequality, it is easy to show that $(Kf)(x)$ exists for all $x \in [c, d]$ whenever $f \in L_2[a, b]$. If $k(x, y)$ is a continuous function on $[c, d] \times [a, b]$, then it is obvious that K satisfies Property (c). Now an important result for the integral operator K satisfying Property (c) is described.

THEOREM 2.2. *If the integral operator K satisfies Property (c), then*

$$N(K)^{\perp} = \vee\{k_x | x \in [c, d]\}.$$

PROOF. Let $M = \{k_x \mid x \in [c, d]\}$ and let $f_0 \in M^{\perp}$. Then for all $x \in [c, d]$, $(f_0, k_x) = 0$. Thus for all $x \in [c, d]$

$$(Kf_0)(x) = \int_a^b k(x, y) f_0(y)\, dy = (k_x, f_0) = 0.$$

The above equality is valid since all functions are assumed to be real valued. It implies that $f_0 \in N(K)$. Thus, $M^{\perp} \subset N(K)$. Hence, it follows that

$$N(K)^{\perp} \subset M^{\perp\perp} = \vee\{k_x \mid x \in [c, d]\}.$$

Conversely, let $f_0 \in N(K)$. Then $(Kf_0)(x) = 0$ for almost all $x \in [c,d]$. Since K satisfies Property (c), $(Kf_0)(x) = 0$ for all $x \in [c,d]$. Then for $x \in [c,d]$,

$$(k_x, f_0) = \int_a^b k_x(y) f_0(y)\, dy = \int_a^b k(x,y) f_0(y)\, dy = (Kf_0)(x) = 0.$$

It follows that $k_x \in N(K)^\perp$ for all $x \in [c,d]$. Thus,

$$\vee\{k_x \mid x \in [c,d]\} \subset N(K)^\perp.$$

Therefore, the equality of this theorem is proved. $\diamond$

Recall that f_0 is the minimum norm solution to $Kf = g$ if and only if $Kf_0 = g$ and $f_0 \in N(K)^\perp$. Therefore, Theorem 2.2 tells us that f_0 can be approximated as closely as desired by (finite) linear combinations of the k_x's.

For given n points $x_1, x_2, \cdots, x_n$, the next theorem shows how to choose constants $c_1, c_2, \cdots, c_n$ so that $\sum_{j=1}^n c_j k_{x_j}$ will best approximate the minimum norm solution f_0 to $Kf = g$ in the L_2-norm without knowing f_0, that is, only $g(x)$ and the kernel $k(x,y)$ need to be known to determine the c_j's. Most expansion methods for solving the first kind integral equations pick the constants $c_1, c_2, \cdots, c_n$ to minimize the norm of the residual instead of the norm difference from the desired solution. Let $\vec{c} = (c_1, c_2, \cdots, c_n)^T$, $\vec{b} = (g(x_1), g(x_2), \cdots, g(x_n))^T$ and let A be the $n \times n$ matrix whose ij-th component is (k_{x_j}, k_{x_i}). With this notation, the following theorem is obtained.

THEOREM 2.3. *If f_0 is the minimum norm solution to the first kind integral equation $Kf = g$ with Property (c), then the minimum of $\| f_0 - \sum_{j=1}^n c_j k_{x_j} \|$ over all constants $c_1, c_2, \cdots, c_n$ occurs when $\vec{c}$*

is a solution of $A\vec{x} = \vec{b}$. *Moreover, the value of* $f_n = \sum_{j=1}^{n} c_j k_{x_j}$ *is independent of which solution* $\vec{c}$ *of* $A\vec{x} = \vec{b}$ *one takes.*

PROOF. By an important property of Hilbert spaces, $\| f_0 - \sum_{j=1}^{n} c_j k_{x_j} \|$ is minimized when the c_j's are chosen so that $\sum_{j=1}^{n} c_j k_{x_j}$ is the orthogonal projection of f_0 onto $\vee\{k_{x_1}, k_{x_2}, \cdots, k_{x_n}\}$. Hence for each i,

$$(f_0 - \sum_{j=1}^{n} c_j k_{x_j}, k_{x_i}) = 0, \text{ i.e., } (f_0, k_{x_i}) = \sum_{j=1}^{n} (k_{x_j}, k_{x_i}) c_j.$$

On the other hand,

$$(f_0, k_{x_i}) = \int_a^b f_0(y) k_{x_i}(y)\, dy = \int_a^b k(x_i, y) f_0(y)\, dy = g(x_i).$$

This proves the first part.

The second part follows immediately from the uniqueness of the orthogonal projection $f_n = \sum_{j=1}^{n} c_j k_{x_j}$ of f_0 onto $\vee\{k_{x_1}, k_{x_2}, \cdots, k_{x_n}\}$.

In Theorem 2.3, if k_{x_j}'s are linearly independent, then A is positive definite and hence the unique solution to $A\vec{x} = \vec{b}$ can b found numerically using the Cholesky factorization. But there are often some cases where k_{x_j}'s are linearly dependent. In this case, Theorem 2.3 shows that any solution to $A\vec{x} = \vec{b}$ can be used to form an approximate solution $\sum_{j=1}^{n} c_j k_{x_j}$ of f_0. Therefore, the minimum solution, $\vec{c} = A^{\dagger}\vec{b}$, to the linear system $A\vec{x} = \vec{b}$ can be used to find an apprixmate solution f_n of the form $\sum_{j=1}^{n} c_j k_{x_j}$. Recall that if $A = U\Sigma V^T$ is the singular value decomposition of A, then $\vec{c} = A^{\dagger}\vec{b} = V\Sigma^{\dagger}U^T\vec{b}$, where $\Sigma^{\dagger}$ is the diagonal matrix obtained by taking the reciprocals of the nonzero elements in the diagonal matrix Σ [5].

In what follows, it will be considered how given n, an optimal set of n points $x_1, x_2, \ldots, x_n$ can be chosen so that $\sum_{j=1}^{n} c_j k_{x_j}$ can best

approximate the minimum norm solution, f_0, of $Kf = g$. Let $f_n = \sum_{j=1}^{n} c_j k_{x_j}$ be the approximate solution obtained by Theorem 2.3 for the minimum norm solution f_0 to $Kf = g$. Then, it is easy to show that

$$\|f_0 - f_n\|^2 = \|f_0\|^2 - \|f_n\|^2 \text{ and } \|f_n\|^2 = \sum_{i=1}^{n} c_i g(x_i).$$

By virture of Theorem 2.3 and the above argument, the following theorem which provides a **numerical method** can be easily stated.

THEOREM 2.4. *Let f_0 be the minimum norm solution to $Kf = g$ and let n be a fixed number. Then, to minimize $\|f_0 - \sum_{j=1}^{n} c_j k_{x_j}\|$ over all $c_1, c_2, \ldots, c_n$ and $x_1, x_2, \ldots, x_n$, one need only solve : minimize $-\sum_{j=1}^{n} c_j g(x_j)$ over all $x_1, x_2, \ldots, x_n$, where $(c_1, c_2, \ldots, c_n)^T = A^{\dagger}\vec{b}$ and $\vec{b} = (g(x_1), g(x_2), \ldots, g(x_n))^T$.*

Notice that $\|f_0\|^2$ being unknown does not inhibit solving this minimization problem. Since $-\|f_n\|^2 = -\sum_{j=1}^{n} c_j g(x_j)$ in Theorem 2.4 can be thought of as a function of n variables, $x_1, x_2, \cdots, x_n$, we have a problem of minimizing the nonlinear function $F(x_1, x_2, \cdots, x_n) = -\sum_{j=1}^{n} c_j g(x_j)$ subject to $c \leq x_i \leq d$ $(i = 1, 2, \cdots, n)$. This minimization problem can be solved numerically by a subroutine E04JAF in the NAG (Numerical Algorithms Group) library. Once n is given, this numerical method provides an optimal set of n points and n basis functions to approximate the minimum norm solution f_0 of $Kf = g$. Since an optimal number of n yielding the "best" possible approximate solution is not known in advance, the numerical method is repeated starting from $n = 1$ until certain criteria are satisfied. The criteria used in this algorithm are $\| g - Kf_n \| < 10^{-13}$ or $(\| f_n \|^2 - \| f_{n-1} \|^2) < 10^{-13}$. Here, the condition $(\| f_n \|^2 - \| f_{n-1} \|^2) < 10^{-13}$ means that f_n is almost as close to f_0 as f_{n-1} on computers with about 15

decimal digits of accuracy, in other words, more iterations exceeding this n may not produce the better approximate solution to f_0. The tolerance 10^{-13} was found on the basis of numerical experiments.

3. Perturbation Results.

From now on, let $P_{n,\bar{x}}$ denote an orthogonal projection of $N(K)^{\perp}$ onto $\vee\{k_{x_1}, k_{x_2}, \cdots, k_{x_n}\}$, where $x_1, x_2, \cdots, x_n$ are n points in the interval $[c, d]$. Let f_0 and f_0^{ϵ} denote the minimum norm solutions to $Kf = g$ and $Kf = g^{\epsilon}$ respectively. In the previous section, for the compact integral operator K satisfying Property (c) it was shown that $N(K)^{\perp} = \vee\{k_x \mid c \leq x \leq d\}$ and the "best" approximate solution f_n on $\vee\{k_{x_1}, k_{x_2}, \cdots, k_{x_n}\}$ for the minimum norm solution f_0 to $Kf = g$ is given by $f_n = P_{n,\bar{x}} f_0$, see Theorem 2.3. From Theorem 2.2, f_n can approximate f_0 arbitrarily closely in the L_2-norm by an appropriate choice of n and $x_1, x_2, \cdots, x_n$.

The problem $Kf = g$ is said to be *well-posed* if the restriction of K to $N(K)^{\perp}$ has a continuous inverse on $R(K)$, i.e., $K|_{N(K)^{\perp}} : N(K)^{\perp} \to R(K)$ has a continuous inverse, i.e., there exists a constant $C > 0$ such that $\| f_0 - f_0^{\epsilon} \| \leq C \| g - g^{\epsilon} \|$ for all g, $g^{\epsilon} \in R(K)$ [6]. The problem $Kf = g$ is called *ill-posed* if $Kf = g$ is not well-posed. Since it is well-known that the problem of solving first kind integral equations is ill-posed, two new definitions are introduced below for the purpose of seeing how serious the ill-posedness of the problem is. All subspaces mentioned in this section are assumed to be closed sets.

DEFINITION 3.1. The problem $Kf = g$ is *stable* on a subspace M of $N(K)^{\perp}$ if there exists $C > 0$ such that $\| f_s - f_s^{\epsilon} \| \leq C \| g - g^{\epsilon} \|$ for all g, $g^{\epsilon} \in R(K)$, where $f_s = Pf_0$, $f_s^{\epsilon} = Pf_0^{\epsilon}$, and P is the orthogonal projection of $N(K)^{\perp}$ onto M. Here, f_0 and f_0^{ϵ} are the minimum norm solutions to $Kf = g$ and $Kf = g^{\epsilon}$ respectively.

DEFINITION 3.2. The problem $Kf = g$ is *well-posed on a subspace*

M of $N(K)^{\perp}$ if there exists $C > 0$ such that $\| f_0 - f_0^{\epsilon} \| \leq C \| g - g^{\epsilon} \|$ for all $g,\ g^{\epsilon} \in K(M)$.

Note that if $Kf = g$ is well-posed, then $Kf = g$ is well-posed on every subspace of $N(K)^{\perp}$. Hence, $Kf = g$ is well-posed on $N(K)^{\perp}$ if and only if $Kf = g$ is well-posed. It is also clear that $Kf = g$ is stable on $N(K)^{\perp}$ if and only if the problem $Kf = g$ is well-posed. It was shown in [4] that bounded linear operators with closed range have continuous Moore-Penrose generalized inverses. Thus, $Kf = g$ is well-posed when $R(K)$ is closed.

THEOREM 3.3. *If M is a subspace of $N(K)^{\perp}$ such that $K(M)$ is closed, then K is well-posed on M.*

PROOF. Define $K_m = K|_M : M \to L_2[c,d]$. Then, K_m is a bounded linear operator with closed range and hence $\| K_m^{\dagger} \|$ is bounded. Let $Kf_0 = g,\ Kf_0^{\epsilon} = g^{\epsilon} \in K(M)$, where $f_0,\ f_0^{\epsilon} \in M$. Then $f_0 = K_m^{\dagger} g$ and $f_0^{\epsilon} = K_m^{\dagger} g^{\epsilon}$ and hence

$$\| f_0^{\epsilon} - f_0 \| = \| K_m^{\dagger} g^{\epsilon} - K_m^{\dagger} g \| \leq \| K_m^{\dagger} \| \, \| g^{\epsilon} - g \| .$$

Thus K is well-posed on M. $\diamond$

COROLLARY 3.4. *If K satisfies Property (c), then $Kf = g$ is well-posed on $\vee\{k_{x_1}, k_{x_2}, \cdots, k_{x_n}\}$.*

Let T be a compact integral operator from $L_2[a,b]$ into $L_2[a,b]$. Recall that a subspace M reduces T if $T(M) \subset M$ and $T(M^{\perp}) \subset M^{\perp}$.

THEOREM 3.5. *If M is a subspace of $N(T)^{\perp}$ such that $T(M)$ is closed and M reduces T, then $Tf = g$ is stable on M.*

To prove this theorem, define the operator $T_m = T|_M : M \to L_2[a,b]$. Let P be the orthogonal projection of $L_2[a,b]$ onto M, and Q be the orthogonal projection of $L_2[a,b]$ onto the closed subspace $R(T_m) = T(M)$. Define $S = T_m^{\dagger} Q T : L_2[a,b] \to M$.

LEMMA 3.6. $S = P$.

PROOF. Since $L_2[a,b] = N(T) \oplus N(T)^{\perp} = N(T) \oplus M \oplus (M^{\perp} \cap N(T)^{\perp})$, for each $h \in L_2[a,b]$ $h = h_1 + h_2 + h_3$, where $h_1 \in N(T), h_2 \in M$ and $h_3 \in M^{\perp} \cap N(T)^{\perp}$. Since $Ph = h_2$ for each $h \in L_2[a,b]$, it is sufficient to show that $Sh = h_2$ for each $h \in L_2[a,b]$. In fact,

$$\begin{aligned} Sh = (T_m^{\dagger} QT)(h) \qquad &= (T_m^{\dagger} QT)(h_1 + h_2 + h_3) \\ = (T_m^{\dagger} Q)(Th_2 + Th_3) &= T_m^{\dagger}(QTh_2 + QTh_3). \end{aligned}$$

Since M reduces T, $T(M^{\perp}) \subset M^{\perp} \subset (T(M))^{\perp}$ and hence $QTh_3 = 0$ and $QTh_2 = Th_2$. Thus, one obtains

$$Sh = T_m^{\dagger} Th_2 = T_m^{\dagger} T_m h_2 = h_2.$$

This last equality holds because $M \subset N(T)^{\perp}$ implies $N(T_m) = \{0\}$ and hence $T_m^{\dagger} T_m f = f$ for all $f \in M$.

PROOF OF THEOREM 3.5. Let g and g^{ϵ} be in $R(T)$, and let f_0 and f_0^{ϵ} be the minimum norm solutions to $Tf = g$ and $Tf = g^{\epsilon}$, respectively. From the definition of $T_m^{\dagger}$, it follows that $T_m^{\dagger} g = T_m^{\dagger} Qg$. From Lemma 3.6, $S = P$ and hence

$$T_m^{\dagger} g = T_m^{\dagger} Qg = T_m^{\dagger} QTf_0 = Sf_0 = Pf_0,$$

$$T_m^{\dagger} g^{\epsilon} = T_m^{\dagger} Qg^{\epsilon} = T_m^{\dagger} QTf_0^{\epsilon} = Sf_0^{\epsilon} = Pf_0^{\epsilon}.$$

Then, it is obtained

$$\| f_s - f_s^{\epsilon} \| = \| Pf_0 - Pf_0^{\epsilon} \| = \| T_m^{\dagger} g - T_m^{\dagger} g^{\epsilon} \| \leq \| T_m^{\dagger} \| \, \| g - g^{\epsilon} \| .$$

Since $T(M)$ is closed, $\| T_m^{\dagger} \|$ is bounded. Thus, $Tf = g$ is stable on M.

Since invariant subspaces of a self-adjoint operator are reducing, the reducing subspace hypothesis on M in the above theorem can be weakened to M only being an invariant subspace when T is self-adjoint, i.e., $T^* = T$. The following theorem provides an upper bound for $\| f_n - f_n^\epsilon \|$ when K satisfies Property (c).

THEOREM 3.7. *Let K satisfy Property (c) and let λ_1 be the smallest nonzero singular value of A. Then for all g, $g^\epsilon \in R(K)$*

$$\| f_n - f_n^\epsilon \| \leq \frac{\| \vec{b} - \vec{b}^\epsilon \|}{\sqrt{\lambda_1}},$$

where $\vec{b} = (g(x_i))_{i=1}^n$, $\vec{b}^\epsilon = (g^\epsilon(x_i))_{i=1}^n$, $f_n = P_{n,\vec{x}} f_0$, $f_n^\epsilon = P_{n,\vec{x}} f_0^\epsilon$, and $A = (a_{ij})_{i,j=1}^n$ and $a_{ij} = (k_{x_j}, k_{x_i})$.

PROOF. Let g and g^ϵ be in $R(K)$. Then one obtains

$$\begin{aligned} \|f_n - f_n^\epsilon\|^2 &= \|P_{n,\vec{x}}(f_0 - f_0^\epsilon)\|^2 \\ &= \sum_{i=1}^n (A^\dagger(\vec{b} - \vec{b}^\epsilon))_i (\vec{b} - \vec{b}^\epsilon)_i \\ &= (A^\dagger(\vec{b} - \vec{b}^\epsilon), \vec{b} - \vec{b}^\epsilon) \\ &\leq \|A^\dagger\| \, \|\vec{b} - \vec{b}^\epsilon\|^2, \end{aligned}$$

where the last inequality is the Schwarz inequality on R^n. Since A is a symmetric, positive semi-definite matrix, the singular values of A are the eigenvalues of A and hence $\|A^\dagger\| = \frac{1}{\lambda_1}$. Therefore, the proof is complete.

To prove the following theorem, let $M = \vee\{k_{x_1}, k_{x_2}, \ldots, k_{x_n}\}$ and let A be an $n \times n$ matrix defined as in the above theorem.

THEOREM 3.8. *Let λ_1 be the smallest nonzero singular value of A. If K satisfies Property (c), then for every nonzero $f \in L_2[a,b]$*

$$\frac{\|f_n\|}{\|f\|} \leq \min\left\{1, \sqrt{\frac{1}{\lambda_1}\sum_{i=1}^{n}\|k_{x_i}\|^2}\right\}, \quad \text{and}$$

$$\lambda \leq \inf_{\substack{f\in M \\ f\neq 0}} \frac{\sum_{i=1}^{n}|(k_{x_i},f)|^2}{\|f\|^2} \leq \left(\sup_{\substack{\vec{d}\in R(A) \\ \|\vec{d}\|=1}} (\vec{d}, A^{\dagger}\vec{d})\right)^{-1},$$

where $f_n = P_{n,\vec{x}} f_0$ and f_0 is the orthogonal projection of f onto $N(K)^{\perp}$. Moreover, if A is nonsingular, then $\lambda_1 = \inf_{\substack{f\in M \\ f\neq 0}} \frac{\sum_{i=1}^{n}|(k_{x_i},f)|^2}{\|f\|^2}$.

PROOF. Let f be a nonzero function in $L_2[a,b]$. Then, $\|f_n\| \leq \|f_0\| \leq \|f\|$ and hence $\frac{\|f_n\|}{\|f\|} \leq 1$. Put $g = Kf = Kf_0$. Applying Theorem 3.7 for this $g \in R(K)$, one obtains

$$\begin{aligned}\| f_n\|^2 \leq \| A^{\dagger} \| \sum_{i=1}^{n} |g(x_i)|^2 &= \| A^{\dagger} \| \sum_{i=1}^{n} |(k_{x_i}, f_0)|^2 \\ = \| A^{\dagger} \| \sum_{i=1}^{n} |(k_{x_i}, f)|^2 &\leq \| A^{\dagger} \| (\sum_{i=1}^{n} \| k_{x_i}\|^2) \| f\|^2.\end{aligned}$$

Since $\| A^{\dagger} \| = \frac{1}{\lambda_1}$, the first inequality of this theorem follows from the above inequalities. For the proof of the second inequality, let $f \in M$. From above, for all $f \in L_2[a,b]$

$$\| f_n \|^2 \leq \| A^{\dagger} \| \sum_{i=1}^{n} |(k_{x_i}, f_0)|^2 = {\lambda_1}^{-1} \sum_{i=1}^{n} |(k_{x_i}, f_0)|^2.$$

Hence $\lambda_1 \leq \frac{\sum_{i=1}^{n}|(k_{x_i},f_0)|^2}{\|f_n\|^2}$. Now let $f = f_0 = f_n \in M$ to obtain

$$\lambda_1 \leq \inf_{\substack{f\in M \\ f\neq 0}} \frac{\sum_{i=1}^{n}|(k_{x_i},f)|^2}{\| f \|^2}.$$

Let $f = \sum_{i=1}^{n} c_i k_{x_i} \in M$ and let $\vec{c} = (c_i) \in R^n$. Then

$$\inf_{\substack{f \in M \\ f \neq 0}} \frac{\sum_{i=1}^{n} |(k_{x_i}, f)|^2}{\| f \|^2} = \inf_{\substack{\vec{c} \in R^n \\ (A\vec{c}, \vec{c}) \neq 0}} \frac{(A\vec{c}, A\vec{c})}{(A\vec{c}, \vec{c})} = \left(\sup_{\substack{\vec{c} \in R^n \\ A\vec{c} \neq 0}} \frac{(A\vec{c}, \vec{c})}{(A\vec{c}, A\vec{c})} \right)^{-1}$$

$$\leq \left(\sup_{\substack{\vec{c} \in N(A)^{\perp} \\ A\vec{c} \neq 0}} \frac{(A\vec{c}, \vec{c})}{(A\vec{c}, A\vec{c})} \right)^{-1} = \left(\sup_{\substack{\vec{d} \in R(A) \\ \vec{d} \neq 0}} \frac{(\vec{d}, A^{\dagger}\vec{d})}{(\vec{d}, \vec{d})} \right)^{-1}$$

$$= \left(\sup_{\substack{\vec{d} \in R(A) \\ \|\vec{d}\| = 1}} (\vec{d}, A^{\dagger}\vec{d}) \right)^{-1}.$$

Hence, the second inequality is proved. If A is nonsingular, then $R(A) = R^n$ and hence from above

$$\left(\sup_{\substack{\vec{d} \in R(A) \\ \|\vec{d}\| = 1}} (\vec{d}, A^{\dagger}\vec{d}) \right)^{-1} = \frac{1}{\| A^{\dagger} \|} = \lambda_1.$$

This completes the proof.

Corollary 3.4 shows that if the compact operator K satisfies Property (c), then $Kf = g$ is well-posed on $M = \vee\{k_{x_1}, k_{x_2}, \cdots, k_{x_n}\}$. The next example shows that $Kf = g$ is not necessarily stable on M.

EXAMPLE 3.9. Let $K : L_2[0,1] \to L_2[0,1]$ be an integral operator with the kernel $k(x,y)$ given by

$$k(x,y) = \begin{cases} 1, & \text{if } y \leq x; \\ 0, & \text{otherwise.} \end{cases}$$

Since $(Kf)(x) = \int_0^x f(y)\,dy$ is continuous on $[0,1]$ for $f \in L_2[0,1]$, K clearly satisfies Property (c). It can be easily shown that K is a compact operator with $N(K) = \{0\}$. Thus if $g(0) = 0$, $g'(x)$ exists on $[0,1]$ and $g' \in L_2[0,1]$, then the integral equation $(Kf)(x) = g(x)$, $0 \leq x \leq 1$, has the minimum norm solution $g' \in L_2[0,1]$ which belongs to $N(K)^{\perp}$. Put $g(x) = x$ and $g^{\epsilon}(x) = x + x^{1/\epsilon}$, where $0 < \epsilon < 1$. Clearly g and $g^{\epsilon} \in R(K)$. Let $n = 1$ and $M = \vee\{k_1\}$. Then $f_0 = 1$, $f_0^{\epsilon} = 1 + \frac{1}{\epsilon}x^{1/\epsilon - 1}$, $\| g^{\epsilon} - g\|^2 = \frac{\epsilon}{2+\epsilon} \to 0$ as $\epsilon \to 0$, and

$$\begin{aligned} f_n - f_n^{\epsilon} = P_n(f_0 - f_0^{\epsilon}) \qquad &= P_n\left(\frac{1}{\epsilon}y^{1/\epsilon - 1}\right) \\ = \frac{1}{\epsilon}\left(\frac{(y^{1/\epsilon-1}, k_1)}{(k_1, k_1)}k_1\right) &= k_1. \end{aligned}$$

Hence $\| f_n - f_n^{\epsilon} \| = 1$ for all $0 < \epsilon < 1$.

Notice that in the above example $\| \vec{b} - \vec{b^{\epsilon}} \| = 1$ even if $\lim_{\epsilon \to 0} \| g - g^{\epsilon} \| = 0$. Thus, Theorem 3.7 tells us that $\| f_n - f_n^{\epsilon} \|$ may not converge to 0. From Example 3.9, it can be easily shown that $k_1 \notin R(K^*)$, i.e., $k_1 \in \overline{R(K^*)} - R(K^*)$.

THEOREM 3.10. *If $K : L_2[a,b] \to L_2[c,d]$ satisfies Property (c) and $M = \vee\{k_{x_1}, k_{x_2}, \cdots, k_{x_n}\} \subset R(K^*)$, then $Kf = g$ is stable on M.*

PROOF. Let g, $g^{\epsilon} \in R(K)$, and let $\vec{b} = (g(x_i))$ and $\vec{b^{\epsilon}} = (g^{\epsilon}(x_i))$. Since $k_{x_i} \in R(K^*)$ for $i = 1, 2, \cdots, n$, there exists an $h_i \in L_2[c,d]$ such that $K^*h_i = k_{x_i}$ for each i. Hence, one obtains

$$\begin{aligned} |(\vec{b} - \vec{b^{\epsilon}})_i| = |g(x_i) - g^{\epsilon}(x_i)| &= |(Kf_0)(x_i) - (Kf_0^{\epsilon})(x_i)| \\ = |(k_{x_i}, f_0 - f_0^{\epsilon})| &= |(h_i, K(f_0 - f_0^{\epsilon}))| \\ = |(h_i, g - g^{\epsilon})| \quad &\leq \| h_i \| \| g - g^{\epsilon} \|. \end{aligned}$$

Using Theorem 3.7, it is obtained

$$\begin{aligned} \| f_n - f_n^\epsilon \| &\leq \frac{\| b - b^\epsilon \|}{\sqrt{\lambda_1}} \\ &\leq \sqrt{\frac{n}{\lambda_1}} \max_{1 \leq i \leq n} |b_i - b_i^\epsilon| \\ &\leq C\sqrt{\frac{n}{\lambda_1}} \| g - g^\epsilon \|, \end{aligned}$$

where $C = \max\{\| h_i \| : 1 \leq i \leq n\}$ and λ_1 is the smallest nonzero singular value of the matrix A. Hence $Kf = g$ is stable on M.

4. Numerical Results.

All computer runs were done in double precision on the Hitachi Data Systems (HDS) AS/9160 using the IBM VS FORTRAN compiler and using the MVS operating system. Double precision arithmetic on this machine means about 15 decimal digits of accuracy. The integer n is the number of functions, k_x's, that were used. The minimum norm solution to the $n \times n$ linear system $A\vec{x} = \vec{b}$, see Theorem 2.3 for notation, was obtained using LINPACK [2] subroutine DSVDC. All integral equations tested were of the form $\int_0^1 k(x,y)f(y)\,dy = g(x)$, $0 \leq x \leq 1$. Therefore, the ij-th component of A is

$$a_{ij} = (k_{x_j}, k_{x_i}) = \int_0^1 k(x_j, y)k(x_i, y)\,dy.$$

This integral was calculated by the exact arithmetic formula if possible. Otherwise, it was approximated by breaking the interval $[0,1]$ into m subintervals and applying the Gauss-Legendre 4-point rule to each subinterval. This type of numerical integration method is called the repeated (m-panel) Gauss-Legendre 4-point rule. For the purpose of evaluating the algorithm, the minimum norm solution, $f_0(y)$, was

entered into the program as a function statement and $b_i = g(x_i)$ was then calculated by

$$g(x_i) = \int_0^1 k(x_i, y) f(y)\, dy,$$

where $f(y)$ is some given function used to determine $g(x)$ and may be different from $f_0(y)$. The above integral used to compute $g(x_i)$ was calculated by the exact arithmetic formula if possible, otherwise it was approximated by the repeated ($3m$-panel) Gauss-Legendre 4-point rule. $3m$ panels were used instead of m panels in order to get more accurate calculation for $g(x_i)$. Once an approximate solution $f_n(y) = \sum_{j=1}^{n} c_j k_{x_j}(y)$ for f_0 has been calculated, three different kinds of errors - L2NM, AE and RE - between $f_0(y)$ and $f_n(y) = \sum_{j=1}^{n} c_j k_{x_j}(y)$ were approximated. Specifically, L2NM is an approximation to $\| f_0 - f_n \|_2$ by the repeated (4-panel) Gauss-Legendre 4-point rule, AE $= \max\{|f_0(y) - f_n(y)| \mid y \in S\}$, and RE $= \max\{|f_0(y) - f_n(y)| \,/\, |f_0(y)| \mid y \in S,\ f_0(y) \neq 0\}$, where $S = \{\frac{n}{100} \mid n = 0, 1, \cdots, 100\}$. RES denotes an approximation to the residual given by $\max\{|(Kf_n)(x) - g(x)| \mid x \in S\}$. The Fortran code for this numerical method is available from the author upon request. The following example describes a special property for the integral operators with smooth kernels.

EXAMPLE 4.1. Let $K : L_2[0,1] \to L_2[0,1]$ be defined by $\int_0^1 k(x,y) f(y)\, dy$, $0 \leq x \leq 1$, where $f \in L_2[0,1]$ and $k(x,y) = e^{xy}$, $\sin(xy)$, or $\cos(xy)$. Then $N(K) = \{0\}$, that is, $N(K)^{\perp} = L_2[0,1]$. To show this, first consider the integral operator K with the kernel $k(x,y) = \sin(xy)$. Suppose that $f \in N(K)$. Since $\sin(xy)$ is continuous,

$$(Kf)(x) = \int_0^1 \sin(xy) f(y)\, dy = 0 \text{ for all } x \in [0,1].$$

With the help of the real analysis on integration, one obtains for each $x \in [0,1]$

$$\frac{d}{dx}\int_0^1 \sin(xy)f(y)\,dy = \int_0^1 \cos(xy)yf(y)\,dy = 0.$$

It follows that $\int_0^1 yf(y)\,dy = 0$ for any $f \in N(K)$ and hence $y \in N(K)^\perp$. Differentiating the integral equation $(Kf)(x) = 0$, $0 \leq x \leq 1$, with respect to x $(2n-1)$-times, where $n = 2,3,\cdots$, one obtains $y^{2n-1} \in N(K)^\perp$ for $n = 2,3,\cdots$. Therefore, $\vee\{y, y^3, y^5, \cdots\} \subset N(K)^\perp$. Since $\sum_{n=1}^{\infty}\frac{1}{2n-1} \geq \sum_{n=1}^{\infty}\frac{1}{2n} = \infty$, by Muntz's Theorem in [1] the linear span of $\{y, y^3, y^5, \cdots\}$ is dense in $L_2[0,1]$. Therefore, $N(K)^\perp = L_2[0,1]$. For the kernel $k(x,y) = \cos(xy)$ or e^{xy}, it can be easily shown in a similar manner as was done for the kernel $k(x,y) = \sin(xy)$.

Since the integral operator $K : L_2[0,1] \to L_2[0,1]$ with e^{xy}, $\sin(xy)$, or $\cos(xy)$ as its kernel satisfies Property (c), by Theorem 2.2 and Example 4.1 $L_2[0,1] = \vee\{e^{\alpha y} \mid 0 \leq \alpha \leq 1\} = \vee\{\cos(\alpha y) \mid 0 \leq \alpha \leq 1\} = \vee\{\sin(\alpha y) \mid 0 \leq \alpha \leq 1\}$. Since $N(K)^\perp = L_2[0,1]$, any solution in $L_2[0,1]$ to $Kf = g$ is the unique minimum norm solution to $Kf = g$ and $R(K)$ is infinite dimensional.

For all the examples given below, n initial points $x_1, x_2, \cdots, x_n$ in the interval $[0,1]$ are chosen by the formula $x_i = \frac{i}{n}$ or $x_i = \frac{i}{n+1}$ for $i = 1, \cdots, n$. It is observed that there is a small difference between these two choices in some of test problems. For this case, the better results are reported. EX denotes the number of test problems, $k(x,y)$ denotes the kernel function, and N_{opt} denotes an optimal number of basis functions that yields the "best" possible approximate solution. Table 1 contains numerical results for the problems for which $R(K)$

is finite dimensional, Table 2 contains numerical results for the case where $R(K)$ is infinite dimensional, and Table 3 contains numerical results for problems with a Volterra kernel and a Green's function kernel.

Table 1 : Numerical results for EX 1 to EX 5

EX	1	2	3	4	5
$f(y)$	e^y	y^2+2y-1	$3y+2$	$\sin(y)$	$(1+y)/4$
$f_0(y)$	$(18-6e)y$ $+4e-10$	$f(y)$	$f(y)$	$f(y)$	$f(y)$
$k(x,y)$	$x-y$	$(x-y)^2$	$(x-y)^3$	$\cos(x-y)$	$1+xy$ if $0 \le x < 1/2$ 1 if $1/2 \le x \le 1$
L2NM	2.3×10^{-15}	6.5×10^{-15}	6.9×10^{-13}	6.9×10^{-15}	5.3×10^{-16}
AE	2.4×10^{-15}	7.6×10^{-15}	1.2×10^{-13}	7.3×10^{-15}	1.1×10^{-15}
RE	9.5×10^{-16}	3.0×10^{-13}	5.9×10^{-13}	7.2×10^{-13}	2.2×10^{-15}
RES	2.9×10^{-16}	3.6×10^{-16}	2.2×10^{-14}	1.8×10^{-15}	2.8×10^{-16}
N_{opt}	3	4	5	3	3

Table 2 : Numerical results for EX 6 to EX 10

EX	6	7	8	9	10
$f(y)$	$\cos 2y$	$\sin y$	$1-2y$	y	$1+y^2$
$f_0(y)$	$f(y)$	$f(y)$	$f(y)$	$f(y)$	$f(y)$
$k(x,y)$	$\cos(xy)$	e^{xy}	e^{xy}	$\sin(xy)$	$\cos(xy)$
L2NM	3.0×10^{-5}	5.4×10^{-7}	7.2×10^{-6}	3.5×10^{-9}	1.1×10^{-7}
AE	6.2×10^{-5}	1.5×10^{-6}	1.2×10^{-5}	8.0×10^{-9}	2.4×10^{-7}
RE	4.1×10^{-4}	4.5×10^{-5}	7.0×10^{-5}	1.4×10^{-8}	1.8×10^{-7}
RES	4.3×10^{-13}	8.0×10^{-13}	1.7×10^{-12}	3.7×10^{-15}	5.6×10^{-13}
N_{opt}	11	12	10	9	7

Table 3 : Numerical results for EX 11 to EX 13

EX	11	12	13
$f(y)$	y	$-e^y$	$(y-1)^2/2$
$f_0(y)$	$f(y)$	$f(y)$	$f(y)$
$k(x,y)$	$x(1-y)$ if $x \le y$ $y(1-x)$ if $x > y$	$x(1-y)$ if $x \le y$ $y(1-x)$ if $y > x$	$(x-y)$ if $y \le x$ 0 if $y > x$
L2NM	5.9×10^{-8}	2.9×10^{-3}	7.3×10^{-4}
AE	$1.0(1.0 \times 10^{-7})$	9.3×10^{-3}	7.6×10^{-4}
RE	$1.0(1.1 \times 10^{-7})$	8.7×10^{-3}	1.0
RES	1.1×10^{-9}	3.4×10^{-5}	3.0×10^{-7}
N_{opt}	6	16	11

The kernel defined as in EX 11 and 12 is called a Green's function kernel, and the kernel in EX 13 is called a Volterra kernel. In EX 11 , L2NM becomes smaller as n increases, but AE and RE have the constant value 1. This can be justified as follows. Recall that $AE = \max\{|f_0(y)-f_n(y)| \mid y \in S\}$ and the set S contains 1. Since $k_{x_i}(1) = 0$ for any $x_i \in [0,1]$, $f_n(1) = 0$ for any n. However, $f_0(1) = 1$. Hence $AE = 1$ and $RE = 1$ were obtained. If 1 is eliminated from the set S when calculating AE and RE, then the numbers in parentheses were obtained. Hence, EX 11 is a typical example of proving numerically that the convergence in L_2-norm does not imply the convergence in L_∞-norm. In EX 13, RE has the contant value 1. This is because $k_x(y) = 0$ for $y > x$ and so $f_n(y) = 0$ if y is greater than $x_i (i = 1, 2, \ldots, n)$, but $f_0(y)$ is non-zero except for $y = 1$. For the examples given in Table 3, f_n does not approximate f_0 well. One of the reasons might be that each basis function k_x is not well-behaved and the problem $Kf = g$ is not well-posed, so that the minimization process required to get f_n does not work well, see Theorem 2.4. Another reason for this could be that it is so difficult to approximate f_0 by a (finite) linear combination of the k_x's.

5. Conclusions.

An advantage of the numerical method discussed in this paper is that it automatically provides an optimal set of basis functions approximating the minimum norm solution in $N(K)^\perp$. If $R(K)$ is finite dimensional, then any minimum norm solution f_0 can be expressed as a finite linear span of k_x's. Thus, for the case where $R(K)$ is finite dimensional, this numerical method approximates f_0 up to about machine accuracy (i.e., 15 decimal digits of accuracy) for n close to the dimension of $R(K)$, see Table 1. However, if $R(K)$ is infinite dimensional, the accuracy of this method depends on how well one can approximate the desired solution by a (finite) linear combination of

the k_x's and how serious the ill-posedness of the problem is, see Tables 2 and 3. This numerical method is based on the L_2-norm difference (not on the uniform norm), so that it is observed that for some problems it can not approximate the minimum norm solution at certain points within the range of integration, see EX 11.

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