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Combinatorial Proof for the Schur Identity

JAEJIN LEE

0. Introduction.

A good way to determine that an identity containing combinatorial concepts holds is to take two sets of combinatorial objects whose cardinalities are the left-hand side and the right-hand side of the given identity respectively, and to exhibit a bijection between them. Many identities containing combinatorial concepts are proved by algebraic methods using the machinery of higher mathematics rather than combinatorial proofs using the combinatorial concepts contained in the given identities. It is very natural to look for purely combinatorial proofs for such identities because they are defined using combinatorial objects.

In [Sch] Schur introduced and proved the Schur identity

$$\sum \frac{2^{\ell(\lambda)}}{z(\lambda)} = 2, \qquad \text{(See Section 2 for definition)}$$

where the summation is over partitions λ of n into odd parts only. Schur's proof depends on certain symmetric functions called Schur functions which are defined to calculate the projective characters of the symmetric group.

In [Mo1] Morris gave a proof for a more general identity that the Schur identity resulted as a corollary, using the Hall-Littlewood symmetric functions (see [Mo2] and [Mo3]). In this paper we consider a

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bijective proof for the Schur identity. That is, we construct a bijection between two sets whose cardinalities are the left-hand side and the right-hand side of the Schur identity respectively.

In Section 1 we outline the definitions and notation used in this paper. In Section 2, we give Morris' algebraic proof for the Schur identity with a little modification. Combinatorial proof for the Schur identity is given in Section 3.

1.Definitions.

In this section, all the necessary definitions are given. We use standard notation **P, Z** for the set of all positive integers, the ring of integers, respectively.

DEFINITION 1.1. A *partition* λ of a nonnegative integer n is a sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that

(1) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$,
(2) $\sum_{i=1}^{\ell} \lambda_i = n$.

We write $\lambda \vdash n$, or $|\lambda| = n$. We say each term λ_i is a *part* of λ and n is the *weight* of λ. The number of nonzero parts is called the *length* of λ and is written $\ell = \ell(\lambda)$. Let $\mathcal{P}$ be the set of all partitions.

The unique partition of 0 is denoted by $\emptyset$. We sometimes abbreviate the partition λ with the notation $1^{j_1}2^{j_2}\ldots$, where j_i is the number of parts of size i. Sizes which do not appear are omitted and if $j_i = 1$, then it is not written. Thus, a partition $(5,3,2,2,2,1) \vdash 15$ can be written $12^3 35$.

NOTATION 1.2. We denote

$$\mathcal{P}_n = \{\mu \in \mathcal{P} \mid \mu \text{ is a partition of } n\} \quad \text{and}$$
$$OP_n = \{\mu \in \mathcal{P}_n \mid \text{ every part of } \mu \text{ is odd}\}.$$

Definition 1.3. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition. The *Ferrers diagram (shape)* D_λ of λ is the array of cells or boxes arranged in rows and columns, λ_1 in the first row, λ_2 in the second row, etc., with each row left-justified. That is,

$$D_\lambda = \{(i,j) \in \mathbf{Z}^2 \mid 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\},$$

where we regard the elements of D_λ as a collection of boxes in the plane with matrix-style coordinates. Sometimes we identify a partition with its diagram, so that $x \in \lambda$ should be interpreted as $x \in D_\lambda$. See *Figure 1.1* for a Ferrers diagram of $(5,4,4,2,1) \vdash 16$.

Definition 1.4. Let λ be a partition and $x = (i,j)$ be a specific cell in the shape D_λ. Let

$$h_x = \{(i,j)\} \cup \{(i,j') \in D_\lambda \mid j' > j\} \cup \{(i',j) \in D_\lambda \mid i' > i\}.$$

We say h_x is the ***hook*** associated with x and $h(x) = |h_x|$ is the ***hook length*** of λ at $x = (i,j) \in \lambda$. Geometrically, $h(x)$ counts the number of cells directly below or directly to the right of x, including x. *Figure 1.2* gives an example of a hook of length 6.

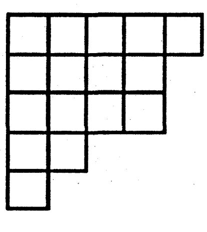

Figure 1.1

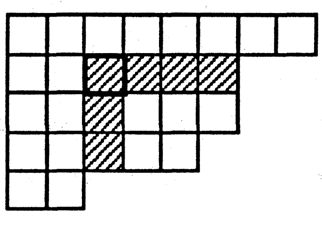

Figure 1.2

DEFINITION 1.5. Let λ be a partition. A *tableau* T of shape λ is an assignment $T : D_\lambda \to \mathbf{P}$ of positive integers to the cells of λ. The *content* of the tableau T, denoted by content(T), is the finite nonnegative vector whose ith component is the number of entries i in T.

DEFINITION 1.6. Let $[n] = \{ 1, 2, \cdots, n \}$. A bijection $\sigma : [n] \to [n]$ is called a *permutation* on $[n]$. We write S_n for the set of all permutations on $[n]$. Any permutation $\sigma \in S_n$ can be written as a product of disjoint cycles. We define a standard representation by requiring that (a) each cycle is written with its largest first, and (b) the cycles are written in increasing order of the largest number. If $\sigma \in S_n$ is written in standard form, define the type of σ, denoted type(σ), to be the sequence $(\rho_1, \rho_2, \cdots, \rho_n)$, where ρ_i is the number of cycles of σ of length i. Note that the number of permutations in S_n of type $\pi = (\rho_1, \rho_2, \cdots, \rho_n)$ is equal to $n!/z(\pi)$, where $z(\pi) = 1^{\rho_1}\rho_1! 2^{\rho_2}\rho_2! \cdots n^{\rho_n}\rho_n!$.

2. Morris' algebraic proof for the Schur identity.

In this section we give an algebraic proof for the Schur identity using the complete symmetric functions and power sums, rather than Hall-Littlewood functions used by Morris. First we introduce the most basic unit in the theory of symmetric functions. See [Mac] or [St] for details.

Let $x_1, x_2, \cdots$ be the infinite variables. Let $\Lambda(x)$, or simply Λ, be the ring of symmetric functions of $x_1, x_2, \cdots$.

DEFINITION 2.1. Let r be a positive integer.

(1) The rth *complete symmetric function* h_r is defined to be

$$h_r = \sum_{i_1 \le i_2 \le \cdots \le i_r} x_{i_1} \cdots x_{i_r}.$$

(2) The rth *power sum* p_r is defined by

$$p_r = \sum_{i\geq 1} x_i^r.$$

By convention, we set $h_0 = p_0 = 1$ and $h_r = p_r = 0$ for $r < 0$. Extend the definition of these symmetric functions to all partitions by

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots ,$$
$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots .$$

Note that $h_1 = p_1 = \sum_i x_i$ and that h_λ and p_λ are all homogeneous of degree $|\lambda|$.

From Definition 2.1 we can find the generating functions for the h_r and p_r.

THEOREM 2.2. [Mac]

(1) $H(t) = \sum_{r\geq 0} h_r t^r = \prod_{i\geq 1} \frac{1}{1 - x_i t}$.

(2) $P(t) = \sum_{r\geq 1} p_r t^{r-1} = \frac{H'(t)}{H(t)}$.

THEOREM 2.3. *Let λ be a partition and let $m_i = m_i(\lambda)$ be the number of parts of λ equal to i. If $z(\lambda) = \prod_{i\geq 1} i^{m_i} m_i!$, then we have*

$$H(t) = \sum_\lambda z(\lambda)^{-1} p_\lambda t^{|\lambda|},$$

or equivalently

$$h_n = \sum_{\lambda\vdash n} z(\lambda)^{-1} p_\lambda.$$

PROOF. From Theorem 2.2 we have

$$\begin{aligned}
H(t) &= \exp \sum_{r\geq 1} p_r t^r / r \\
&= \prod_{r\geq 1} \exp(p_r t^r / r) \\
&= \prod_{r\geq 1} \sum_{m_r=0}^{\infty} (p_r t^r)^{m_r} / r^{m_r} \cdot m_r! \\
&= \sum_{\lambda} z(\lambda)^{-1} p_\lambda t^{|\lambda|},
\end{aligned}$$

which is required. □

Let r be a positive integer and let now

$$H(t) = \frac{(1-t^r)}{(1-t)^r}.$$

Then

$$\begin{aligned}
P(t) = \sum_{r\geq 1} p_r t^{r-1} &= \frac{d}{dt} \log H(t) \\
&= \frac{-rt^{r-1}}{1-t^r} + \frac{r}{1-t} \\
&= -rt^{r-1}(1 + t^r + t^{2r} + \cdots) + r(1 + t + t^2 + \cdots).
\end{aligned}$$

Hence we have

$$p_n = \begin{cases} 0, & \text{if } n \text{ is divisible by } r \\ r, & \text{otherwise.} \end{cases}$$

On the other hand, note that

$$\begin{aligned}
H(t) = \sum_{r\geq 0} h_r t^r &= \frac{(1-t^r)}{(1-t)^r} \\
&= (1-t^r) \sum_{k=0}^{r} \binom{r+k-1}{k} t^k.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned} h_n &= \text{coefficient of } t^n \text{ in } H(t) \\ &= \binom{n+r-1}{r-1} - \binom{n-1}{r-1}. \end{aligned}$$

Since we have $h_n = \sum_{\lambda \vdash n} z(\lambda)^{-1} p_\lambda$ from Theorem 2.3, we get

$$\binom{n+r-1}{r-1} - \binom{n-1}{r-1} = \sum z(\lambda)^{-1} r^{\ell(\lambda)},$$

where the summation is over all partitions λ of n with parts not divisible by r. This fact is worth stating again as theorem.

THEOREM 2.4. *Let r be a positive integer. Then we have*

$$\binom{n+r-1}{r-1} - \binom{n-1}{r-1} = \sum z(\lambda)^{-1} r^{\ell(\lambda)},$$

where the summation is over all partitions λ of n with parts not divisible by r.

Specializing to $r = 2$ in Theorem 2.4, we obtain the Schur identity as follows:

COROLLARY 2.5.

$$\sum \frac{2^{\ell(\lambda)}}{z(\lambda)} = 2,$$

where the summation is over partitions λ of n into odd parts only.

3. Combinatorial proof for the Schur identity.

In this section we construct a bijection which gives a combinatorial proof for the Schur identity.

DEFINITION 3.1. Suppose X is a set of positive integers. A *permutation tableau* on X is a tableau where each number of X appears exactly once. Clearly the number of permutation tableaux of shape λ on X is $|X|!$.

Let S_X be the set of permutations on X (If $X = \{1, \ldots, n\}$, then $S_X = S_n$). Let $\sigma \in S_X$ and write σ in cycle form, $\sigma = \sigma_1 \sigma_2 \ldots \sigma_m$, where the cycles σ_i are written in increasing order of the largest in the cycle. If $\sigma \in S_X$, then let $\overline{\sigma}$ be a permutation obtained from σ in which each cycle of σ is either barred or unbarred. If $\sigma = (42)(8371)$, $\overline{\sigma}$ is one of $(42)(8371)$, $\overline{(42)}(8371)$, $(42)\overline{(8371)}$ and $\overline{(42)(8371)}$. We use the notation $|\sigma|$ to refer to the unbarred version of any $\sigma \in S_X$; e.g., $|\overline{(21)}(43)| = |\overline{(21)(43)}| = (21)(43)$.

Now let $X = \{a_1 < a_2 < \cdots < a_n\}$ be a set of positive integers. Let

$$\Gamma_n(X) = \{\overline{\sigma} \mid \sigma \in S_X,\ \text{type}(\sigma) \in OP_n\} \quad \text{and}$$
$$\pi_n(X) = \{T \mid T \text{ is a circled permutation tableau of shape } n^1 \text{ on } X\},$$

where a circled permutation tableau is a permutation tableau with the entry in a cell $(1,1)$ either circled or uncircled. Then we have the following bijection from $\Gamma_n(X)$ to $\pi_n(X)$.

THEOREM 3.2. *There is a bijection ϕ from $\Gamma_n(X)$ to $\pi_n(X)$.*

PROOF. We describe the bijection recursively. Suppose $T \in \pi_n(X)$. Find the largest entry a_n in T and let $\alpha = (1,j)$ denote its cell. Some of the steps will split into three cases. These cases are:

Case I: $h(\alpha)$ is odd and $h(\alpha) \neq n$. Case II: $h(\alpha)$ is even. Case III: $h(\alpha)$ is odd and $h(\alpha) = n$.

Step 1.1

Modify T and determine the hook to remove. In Case II, modify T by exchanging a_n with the entry in the cell $(1, j+1)$. Let $\beta = (1, j+1)$. In Cases I and III, no change in T is necessary; simply let $\beta = \alpha$.

Step 1.2

Removal of hook. Remove h_β from T. Call this new permutation tableau T'. Let $a_n, b_1, b_2, \ldots, b_r$ be the entries in T in h_β (read left to right). Then $T' \in \pi_{n-r-1}(Y)$, where $Y = X - \{a_n, b_1, \ldots, b_r\}$. Note that $T' = \emptyset$ in Case III. Finally, let τ be the cycle $(a_n b_1 \ldots b_r)$.

Step 1.3

Recursive step. Recursively construct $\overline{\sigma'} \in \Gamma_{n-r-1}(Y)$ from T'.

Step 1.4

Determine a new permutation. In Case I, let $\overline{\sigma} = \overline{\sigma'}\tau$. In Case II, let $\overline{\sigma} = \overline{\sigma'}\overline{\tau}$. In Case III, let $\overline{\sigma} = \tau$ if h_β has no circle and let $\overline{\sigma} = \overline{\tau}$ if h_β has a circle.

Then $\overline{\sigma} \in \Gamma_n(X)$. This construction can be reversed easily. Suppose we were given $\overline{\sigma} \in \Gamma_n(X)$. Let $|\overline{\sigma}| = \sigma_1\sigma_2\cdots\sigma_k$ and write the last cycle σ_k in $|\overline{\sigma}|$ as $(a_n b_1 \ldots b_r)$. (Recall that a_n is the largest in the cycle.) Let α be the cell $(1, n-r)$ in the Ferrers diagram of shape n^1. Note that $r + 1 = |h_\alpha|$.

Step 2.1

Removal the last cycle. Let $\overline{\sigma'}$ denote the permutation on Y obtained by removing the last cycle from $\overline{\sigma}$, where $Y = X - \{a_n, b_1, \ldots, b_r\}$. Thus, $\overline{\sigma'} \in \Gamma_{n-r-1}(Y)$.

Step 2.2

Recursive step. Recursively construct $T' \in \pi_{n-r-1}(Y)$ from $\overline{\sigma'}$.

Step 2.3.

Construction of a new permutation tableau T'' from T'. Note that T' has only one row. Attach h_α filled with entries $a_n, b_1, \ldots, b_r$ (left to right) to the right of T'. The resulting tableau $T'' \in \pi_n(X)$.

Step 2.4

Modify T'' to get T. Finally define T as follows:

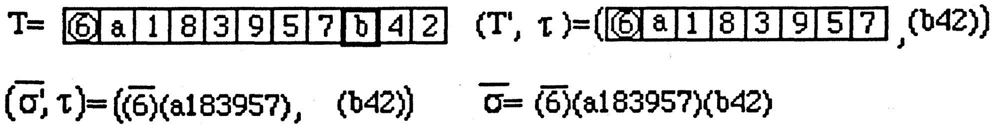

Figure 3.1

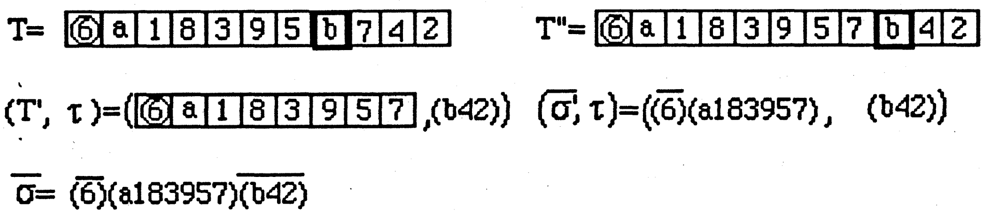

Figure 3.2

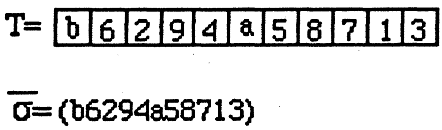

Figure 3.3

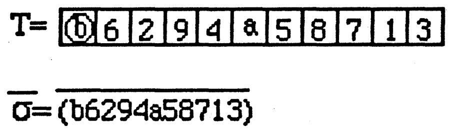

Figure 3.4

(1) If σ_k has no bar on it, let $T = T''$.

(2) If $h(\alpha) \neq n$ and σ_k has a bar on it, T is obtained from T'' by interchanging a_n and the entry in the cell $(1, n - r - 1)$.

(3) If $h(\alpha) = n$ and σ_k has a bar on it, T is obtained from T'' by circling the entry of the cell $(1, 1)$.

It is easy to see that these two constructions are inverses of one other. □

In *Figure 3.1–Figure 3.7*, we give examples of each case in the above description. In these figures, $n = 11$ and we use the alphabet $1 < 2 < \cdots < 9 < a < b$, where $a = 10$ and $b = 11$. In *Figure 3.1*, T is given with α marked. Since $h(\alpha)$ is odd and $h(\alpha) \neq n$, no change in T was made (Step 1.1). Next remove h_α from T to get T' (Step 1.2). The recursive Step 1.3 produces $\overline{\sigma'}$. Finally, the cycle formed from the entries in h_α together with $\overline{\sigma'}$ yields $\overline{\sigma}$ (Step 1.4).

In *Figure 3.2*, note that $h(\alpha)$ is even. Hence we switch the entry "b" in the cell α with the entry "7" in β to get the T''. Now we do the same steps as in *Figure 3.1* except $\overline{\sigma}$ has a bar on its last cycle (Step 1.4).

In *Figure 3.3* and *Figure 3.4*, depending on the circle of the cell $(1,1)$ of h_α, we place a bar on the last cycle.

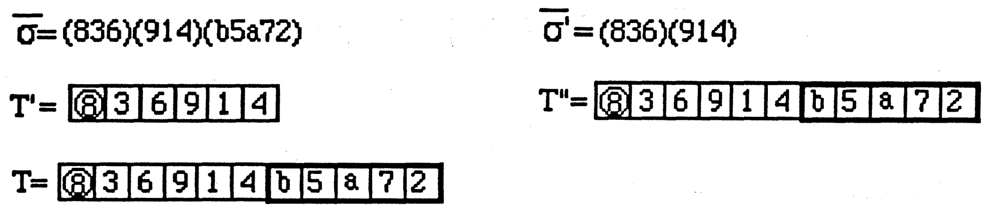

Figure 3.5

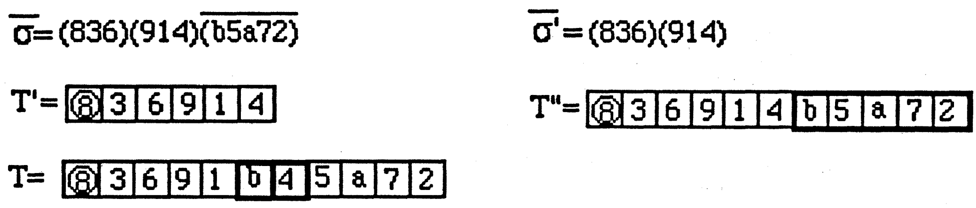

Figure 3.6

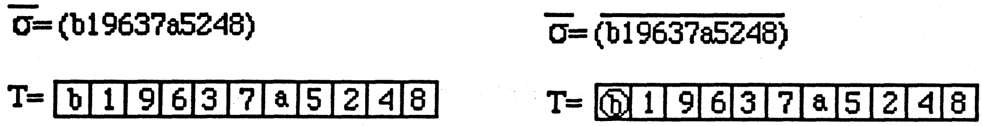

Figure 3.7

In *Figure 3.5*, $\overline{\sigma}$ is given. $\overline{\sigma'}$ is obtained by deleting the last cycle (Step 2.1). Then recursive Step 2.2 produces T'. Next T' is adjusted to make room for the last cycle which is inserted into the hook h_α to give T'' (Step 2.3). Since there is no bar on the last cycle in $\overline{\sigma}$, we have $T = T''$ (Step 2.4). *Figure 3.6* show example of (2) of the Step 2.4. Since there is a bar on the last cycle in $\overline{\sigma}$, T is obtained from T'' by interchanging b and the entry 4 in the cell $(1,6)$. In *Figure 3.7*, depending on the bar on the cycle, we do circling the entry of the first cell.

COROLLARY 3.3.

$$\sum_{\substack{\mu \in OP_n \\ \mu=(1^{j_1}3^{j_3}5^{j_5}\dots)}} 2^{j_1+j_3+j_5\cdots} \frac{n!}{1^{j_1}3^{j_3}5^{j_5}\dots j_1!j_3!j_5!\dots} = 2\,n!$$

PROOF. In Theorem 3.2,

$$\begin{aligned}
|\Gamma_n| &= \sum_{\substack{\sigma \in S_n \\ type(\sigma) \in OP_n}} 2^{\ell(type(\sigma))} \\
&= \sum_{\mu \in OP_n} 2^{\ell(\mu)} \frac{n!}{z(\mu)} \\
&= \sum_{\substack{\mu \in OP_n \\ \mu=(1^{j_1}3^{j_3}5^{j_5}\dots)}} 2^{j_1+j_3+j_5\cdots} \frac{n!}{1^{j_1}3^{j_3}5^{j_5}\dots j_1!j_3!j_5!\dots},
\end{aligned}$$

while $|\pi_n| = 2n!$. □

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Department of Mathematics
Hallym University
Chunchon, Kangwon-Do 300-702