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A Paradox in Geometrical Probability

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In [1], the following problem is presented, together with a solution.

Arson Problem. In Judge Tak A. Chance's courtroom, the floor consists of square tiles with the design on each being a circle with an inscribed equilateral trangle. A person convicted of arson has his sentence determined by a chance happening. A crime of arson carries a sentence of 10 or 20 years, depending on one's luck. An arsonist is blindfolded and allowed to drop a very thin rod onto one of the tiles. If the distance between the points of intersection of the rod with the circle is greater than the side of the inscribed equilateral triangle, the sentence is only 10 years; otherwise, it is 20 years.

Figure 1

Determine the probability that the arsonist gets a 10 year sentence.

This problem can be solved using *geometrical probability*. Suppose an experiment is such that the set of all outcomes (i.e., the *sample space* of the experiment) can be identified with the points in a geometric region R. Then an event E in the sample space can be identified with a subregion r of R. We will say that an outcome of our experiment is a *success* if it belongs to r and r is the *success region* for our experiment. If p is the probability that event E will occur in a single trial of our experiment, it seems reasonable that

$$p = \frac{\text{length of } r}{\text{length of } R}$$

if R is one-dimensional and

$$p = \frac{\text{area of } r}{\text{area of } R}$$

if R is two-dimensional.

Let's now see how we can solve the *Arson Problem* using geometrical probability. Suppose ρ is the radius of a circle on one of the floor tiles. Our experiment is equivalent to the mathematical experiment of randomly selecting a chord of the circle and inquiring as to the probability that its length exceeds that of a side of an inscribed triangle. How *can* we *identify* the outcomes of our experiment with the points in a geometric region? The approach taken in [1] is to identify each chord with its midpoint. Thus the sample space would be associated with the set of all points inside a circle with radius ρ.

There are, however, other ways in which *we could identify the outcomes* in our sample space with the points in a geometric region. For example, we could choose our random chord by fixing the position of one endpoint in advance at a point A on the circle. (If the randomly chosen chord does not go through this point, then, because of symmetry of the circle, we can rotate the circle until an endpoint of the chord coincides with A.) Then each chord can be identified with its other endpoint. Therefore, the sample space of our experiment, the set of all possible chords of the circle, can be identified with the set of all points on the circle.

Another possible approach would be to note that, by symmetry, we may fix the *direction* of the chord in advance. Consider the diameter of the circle perpendicular to this direction. Then we can identify the set of all possible chords of the circle with the points on this diameter.

We now consider the three solutions to the *Arson Problem* which result from the above three ways of identifying the set of all chords of a circle with a geometric region.

Solution 1. Suppose we identify each chord with its midpoint. Then our sample space will be identified with the set of points inside a circles of radius ρ.

We need to find those points of the circle that constitute a success. Figure 2 shows a circle of radius ρ, an inscribed equilateral triangle of side s, and the line segment OC from O (the circles's center) perpendicular to side AB of the triangle.

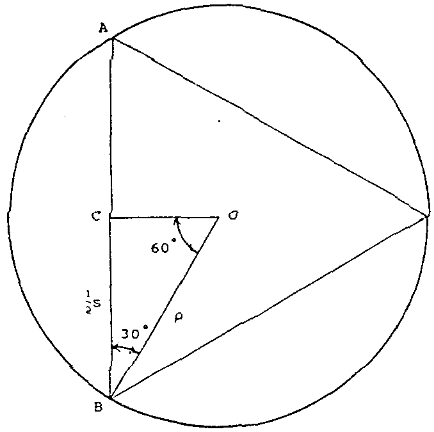

Figure 2

Since the triangle is equilateral, $\angle AOB$ is $120°$. *Other circle poroperties* (Can you name them?) give us $\angle COB = 60°$ and $\overline{AC} = \overline{CB} = \frac{1}{2}s$. Since $\angle OBC = 30°$ (Why?), we have $\overline{OC} = \frac{1}{2}\overline{OB} = \frac{1}{2}\rho$ (Why?)

We know from plane geometry that the greater the distance of a chord from the center of a circle the smaller the chord. Hence the midpoints of the chords whose lengths are greater than s must have their distances from th e center of the circle being less than $\frac{1}{2}\rho$. The success region r and the region R representing the entire sample space are shown in Figure 3.

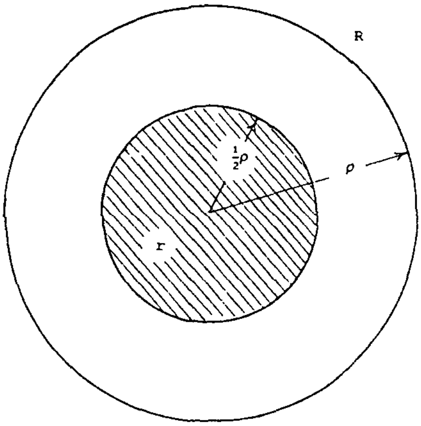

Figure 3

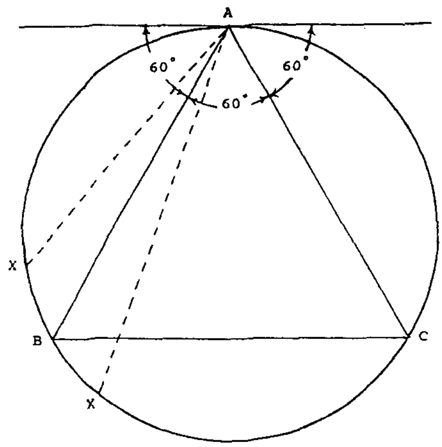

Figure 4

The probability p we want is equal to the area of the circle with radius $\frac{1}{2}\rho$ divided by the area of the circle with radius ρ.

We get

$$p = \frac{\pi\left(\frac{1}{2}\rho\right)^2}{\pi\rho^2} = \frac{1}{4}.$$

Soultion 2. Suppose we fix the position of one end of our chord at vertex A of the inscribed equilateral triangle and then obtain our random chord by randomly choosing the other endpoint X on the circle.

The two angles formed by the tangent to the circle at A and the two sides of the inscribed equilateral triangle with vertex at A have measures of 60° (Why?). A chord through A lying inside $\angle ABC$ has length greater than a side of the triangle. All chords containing A and lying outside $\angle BAC$ have lengths less than or equal to a side of the triangle, thus resulting in failures. We are now in a position to calculate the probability. The other endpoint X of the randomly chosen chord can be anywhere on the circle (the sample space) and will give a success if it belongs to BC. Now length of BC = length of AB = length of AC (Why?).

Thus,

$$p = \frac{\text{length of } BC}{(\text{length of } AB) + (\text{length of } BC) + (\text{length of } AC)}$$

$$= \frac{\text{length of } BC}{3\ (\text{length of } BC)} = \frac{1}{3}$$

Soultion 3. Suppose we choose at random a diameter of the circle by randomly choosing in $[0, 2\pi)$ the angle θ it makes with a fixed diameter. Then each point on our randomly chosen diameter can be associated with the chord perpendicular to the diameter having this point as its midpoint. Therefore each chord of the circle can be identified with an ordered pair $(0, x)$, where x is the distance from a fixed end of the diameter determined by the angle to the midpoint of the chord, as shown in Figure 5.

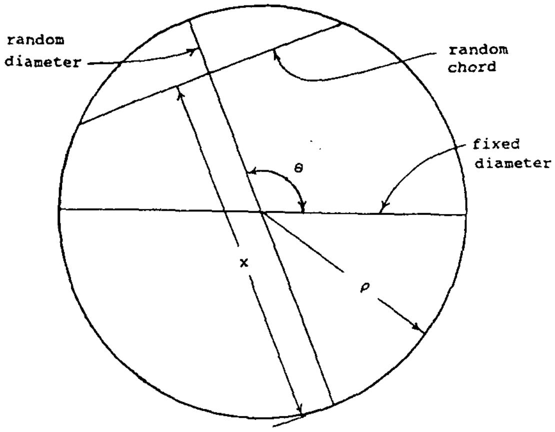

Figure 5.

Thus we see that the sample space of our experiment (the set of all possible chords of the circle) can be identified with the points in the rectangular region

$R = \{(\theta, x) \mid 0 \le \theta < 2\pi,\ 0 < x < 2\rho\}$

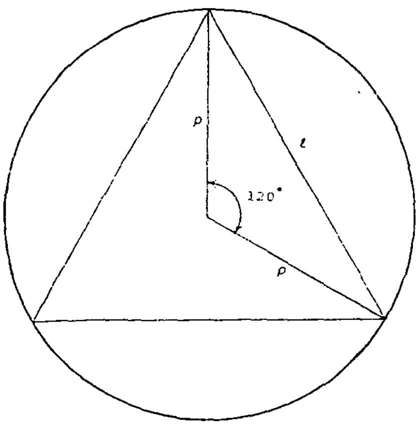

Figure 6

To determine our success region, we first need tofind the length l of a side of an inscribed equilateral triangle.

From figure 6 and the law of coines, we get

$$\begin{aligned} l^2 &= \rho^2 + \rho^2 - 2(\rho)(\rho)\cos 120^\circ \\ &= \rho^2 + \rho^2 - 2\rho^2 \cos 120^\circ \\ &= 2\rho^2 (1 - \cos 120^\circ) \\ &= 2\rho^2 (1 - (-1/2)) \\ &= 2\rho^2 (3/2) = 3\rho^2 \end{aligned}$$

and so

$$l = \rho\sqrt{3}.$$

We see that our random chord's length will be greater than that of a side of an inscribed equilateral triangle if the distance between its midpoint and the center of the circle is less than the distance d between the center of a side of the inscribed equilateral triangle and the center of the circle.

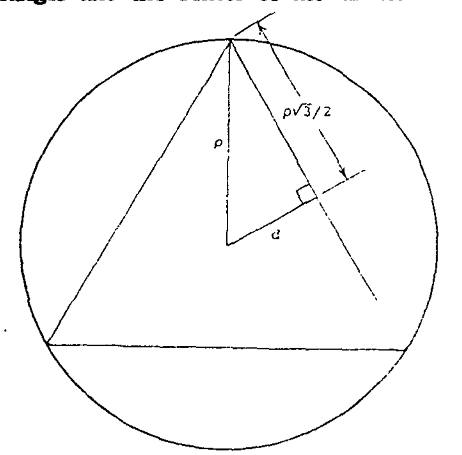

Figure 7.

From Figure 7, we see that

$$\rho^2 = d^2 + (\rho\sqrt{3}/2)^2$$

and therefore

$$d^2 = \rho^2 - (\rho/2)^2 = \rho^2 - 3\rho^2/4 = \rho^2/4,$$

so

$$d = \rho/2.$$

Thus we have a success (i.e., $d > \rho/2$) if $\rho/2 < x < 3\rho/2$. Figure 8 shows the region R representing our sample space together with the shaded success region r.

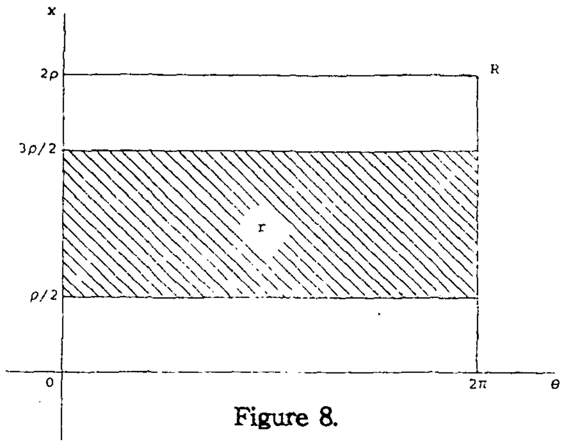

Figure 8.

Therefore the probability p of a success will be

$$p = \frac{\text{area of } r}{\text{area of } R} = \frac{(\rho)(2\pi)}{(2\rho)(2\pi)} = \frac{1}{2}.$$

Observe that each of these three methods of solving the Arson Problem resulted in a different answer. What is wrong? The answer to this question lies in the interpretation of the phrase "randomly choosing a chord of a circle." A different interpretation of this phrase is used in each of the three solutions and thus we have obtained solutions to three different problems rather than one. Which solution isthe most reasonable will depend on how the experiment of choosing a random chord of our circle is actually performed. If we wanted to perform a real-world experiment to randomly choose a chord of our circle, we might do one of the following.

Experiment 1. Let a circlular rod roll along a diameter of our circle. Then the endpoints of the randomly chosen chord will be the two points at which the rod meets the circle.

Experiment 2. A circular disk is spun, held to a flat surface bya pivot point on its edge. Let L be a line in the flat surface through the pivot point. Then the endpoints of the randomly chosen chord will be the points at which L meets the circular edge of the disk.

Experiment 3. A dart is randomly thrown at a circular dartboard. Then the randomly chosen chord will be the unique chord whose midpoint is the point at which the dart hits the dartboard.

Which of our three interpretations of the phrase "randomly choosing a chord of a circle" do you think best fits each of these experiments?

References

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