論 文

Deriving the Fourier Transforms of Pulse Signals Through the Look-up Tables

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찾아보기 목록에 의한 고차펄스의 푸리에 변환법

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Abstract

This paper proposes a novel method for deriving the Fourier transform pairs of high order pulses given in a generalized form. Primarily, modifying the PRS system model, we establish a new model which simplifies the process of Fourier analysis of the n-th order pulse signal, resulting in a representative relationship. In succession, we present the Frame Formula which plays a role of substituent for the parameters in table look-up procedures. Each look-up table contains all the parameters needed to obtain the Fourier transform of the corresponding pulse of any order. Regarding the amount of calculations and the complexity of procedures required to derive the transforms of pulse signals, analytically or numerically, this method is more compact and timesaving than conventional methods. When pulse has a much narrow width or equivalently higher the order of several pulses, the method presented here acts to the best of its true merit.

요 약

본 논문에서는 일반적 형태의 고차원 펄스신호에 대한 푸리에 변환을 구하는 새롭고 용이한 방법을 제안한다. 우선 일반적 형태의 부분응답 전송시스템 모델을 수정하여 n-차 펄스신호의 푸리에 해석을 위한 새로운 모델을 설정하고, 그로부터 차수에 대하여 순환적으로 작용하는 대표 관계식을 유도한다. 이러한 대표 관계식으로부터 목록 찾아보기 방식의 푸리에 변환법을 위한 일반적 뼈대공식을 형성시킨다. 이 때, 각각의 찾아보기 목록은 해당 펄스신호의 모든 차수에 대하여 그 푸리에 변환을 뼈대공식으로 구하기 위해 필요한 모든 매개변수를 포함하게 된다. 고차원 펄스 신호에 대한 푸리에 변환과정의 복잡성과 계산량 등을 비교하여 볼 때, 본 논문에서 제시하는 방법은 기존의 방법에 비하여 대단히 간단하고 계산시간을 단축시킨다. 특별히 몇몇 펄스의 고차 형태는 매우 좁은 폭을 갖게되는데, 이는 최근 지대한 관심을 불러 일으키고 있는 극초단 펄스의 형태를 근사화 하며, 본 논문에서 제시하는 방법은 이들에 대하여 진가를 발휘한다.

**牧園大學校 컴퓨터工學科
論文番號：93－35

I. INTRODUCTION

Pulse signals are often used to describe the system functions or the units of transmission in several communication systems [1]. In recent years, especially, there has been considerable interests in the phenomena of ultrashort pulses [2]. On the other hand, the method of Fourier transform plays an important role in analysis and design of these pulses. Due to the encouraging results of our recent study concerning the cosine-pulses [3], further investigations and generalizations are made throughout this paper.

Firstly, we modify the generalized PRS(partial response signaling) system model for our purposes [4]. In other words, the original model will be endowed with a recursive property by changing the frequency response of the bandlimiting filter following the orders of pulses we want to analyze. After analyzing the modified model and formulating a recursive relationship, we present the Representative Formula which can be used to derive the Fourier transform pairs regardless of pulses of any order becomes issue. However, the modified model does not always provide an accurate solution. Most malformed functions which have some discontinuities in the region of interests fail to have their accurate transforms [5]. In addition, the fact is that periodic functions which contain the corresponding pulses as a part of themselves have to satisfy a sufficient condition in order to obtain accurate transforms from the model. We also present the sufficient condition in a form of truncated convolution integral. However, since most of the pulses used in ordinary transmission systems or in details of short pulse are sufficiently well behaved and are directly applied to the modified model, the condition will be little worth considering the view of this paper.

Although the recursive relationship and the Representative Fomula are advantageous to build up numerical algorithms [1], there seems to be inconveniences when they are employed in deriving analytical solutions. To solve this, we have recently presented a frame formula with a look-up table for deriving the Fourier transforms of the n-th order cosine pulses by the procedure of separation of coefficients [3]. After deeper investigations on the basis of this concept, we develop the Frame Formula in a generalized form which can be applied to all forms of pulses. Several look-up tables for pulses which are frequently used in short pulse transmission are also submitted for analytical and numerical convenience.

The remainder of this paper is organized as follows. In Section II, we describe the problem and the modified PRS system model which will be the focus of our paper. Section III is devoted to the derivation of the Representative Formula from the modified model. In section IV, we develop the main theorem concerning the Frame Formula and submit several look-up tables for pulses of interest. Finally, we draw our concluding remarks in Section V.

II. PROBLEM DESCRIPTION AND THE MODEL

The central problem of this study is to present an easy method for deriving the Fourier transform pairs of pulse signals. The fact is that we can solve this problem through the process of consecutive differentiations or the mthod by convolution theorem, in accordance with the forms of pulses individually [5]. These conventional methods, however, would cause seriously complex and painstaking calculations as the orders of pulses are increased. Realizing this point early, we commenced a study in order to explore an easy method for the subject limited to a specific pulse [3]. This time we plan to elaborate a generalized formula for deriving the Fourier transform of every pulse in a handy method.

First, we define the n-th order pulse as a product of the n-th power of a periodic function and a single rectangular pulse centered at the origin. Then we express the n-th order pulse in time-domain as follows :

$$p_n(t) \equiv A\{v(t)\}^n \Pi\left[\frac{t}{2\tau}\right], \text{ for } n=1,2,3,\ldots \quad (1)$$

where

$$\Pi\left[\frac{t}{2\tau}\right]=\begin{cases}1, & \text{if } |t| \le \tau \\ 0, & \text{elsewhere}\end{cases}$$

and the function $v(t)$ is a periodic function with fundamental period T_0. It is clear that this definition can be used to represent all forms of pulses by changing function $v(t)$ without loss of generality. Deriving the frequency-domain function $P_n(f)$ from the time-domain function $p_n(t)$ will be the focus of this paper. At this point, we can also expect a problem given in the opposite direction. In other words, we are sometimes faced with a problem which requires an inverse transformation from a frequency-domain function given in the form of (1) to the corresponding time-domain function. This exchange in the direction of transformation can make no trouble since the Fourier transform pairs satisfy the duality [5]. Thus there may be no problem even if we only concentrate on the forward direction.

On the other hand, we present a new model in order to obtain a recursive relation for deriving the Fourier transform of the n-th order pulse as shown in Fig.1. Roughly speaking, the model is separated into two parts as in the case of the original PRS system model [4] : The one denoted as $K(f)$ consists of a tapped delay line filter with 2N delay factors and 2N+1 coefficient multipliers which come together in a adder, the other denoted as $G_n(f)$ is an analog filter whose frequency response is equivalent to the overall transfer function of the model for the (n-1)st order. These two filters are connected in cascade with each other and are named as the transversal filter and the bandlimiting filter, respectively.

In order to obtain a recursive formula, we will systematically alter the frequency response of the bandlimiting filter along the order of pulse n, while the structure of the transversal filter will be fixed for all n after decision of its frequency response by the form of given pulse. In addition, the impulse function on the input port of the model has been chosen so that the center of the response always falls on the origin of corresponding domain.

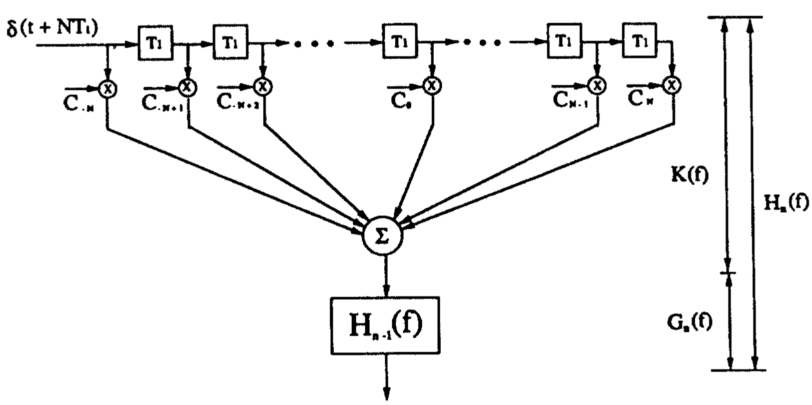


Fig. 1. Modified PRS system Model for the n-th Order Pulses

Ⅲ. DERIVING THE REPRESENTATIVE FORMULA

In this section we wish to derive a generalized formula for the Fourier transforms of pulses after investigating the system functions of the model introduced in section Ⅱ and comparing them with the problem functions. Because the model shown in Fig.1 corresponds to the n-th order pulse and has a recursive property, we have to set an initial function as the strarting point of the recursion. Thus we assume that the frequency response of the bandilimiting filter of the lst order model is the same as that of the minimum-bandwidth PRS system with the exception that the width is not $1/T_1$ but more generally $1/T_2$. In other words, we set

$$G_1(f) = H_0(f) = A\,\Pi(T_2 f) \tag{2}$$

which shapes a Fourier transform pair with

$$g_1(t) = h_0(t) = \frac{A}{T_2}\,\mathrm{sinc}\left[\frac{t}{T_2}\right] \tag{3}$$

where the function 'sinc(·)' is defined as follows :

$$\mathrm{sinc}(x) \equiv \frac{\sin \pi x}{\pi x} \tag{4}$$

From the structure of the model given in Fig.1, we have a Fourier transform pair for the transversal filter

$$k(t) = \sum_{r=-N}^{N} C_r\,\delta(t - rT_1) \Leftrightarrow K(f) = \sum_{r=-N}^{N} C_r \exp(-j2\pi f r T_1) \tag{5}$$

In this relationship, the symbol '⇔' denotes that the two functions on either side of it form a Fourier transform pair. Hence the system functions of the model in both domains are easily obtained by induction starting with the initial functions (2) and (3). They are given in another Fourier transform pair as follows :

$$\begin{aligned} H_n(f) &= K(f)\,H_{n-1}(f) \\ &= \Big[\sum_{r=-N}^{N} C_r \exp(-j2\pi f r T_1)\Big]\,H_{n-1}(f) \\ \Leftrightarrow h_n(f) &= k(t) * h_{n-1}(t) \\ &= \sum_{r=-N}^{N} C_r\,h_{n-1}(t - rT_1), \quad \text{for } n = 1,2,3,\ldots \end{aligned} \tag{6}$$

where the symbol '$*$' stands for the convolution integral of the two functions on either side of it.

It must be noted that the functions in (6) are recursively related to the previous versions of themselves with the initial functions defined in (2) and (3), respectively. Considering the recursive property of the equations in (6) and making vigorous examination of the relationship,we have the following proposition.

<PROPOSITION I>

$$P(n) : H_n(f) = A\Big\{\sum_{r=-N}^{N} C_r \exp(-j2\pi f r T_1)\Big\}^n\,\Pi(T_2 f)$$

$$\Leftrightarrow \quad h_n(t) = \frac{A}{T_2}\Big[\sum_{-N}^{N} \mathbb{C}_r\Big]^n \mathrm{sinc}\left[\frac{t}{T_2} - \frac{T_1}{T_2}\sum_{s=1}^{n} r_s\right]$$

for all $n = 1,2,3,\ldots$

The proof of *PROPOSITION I* is deiscussed in APPENDIX A. In this proposition we have used a new notation which means the depth of summations defined as follows :

$$\Big[\sum \mathbb{C}\Big]^n f(t,\cdot) \equiv \sum_{r_n} C_{r_n} \sum_{r_{n-1}} C_{r_{n-1}} \cdots \sum_{r_1} C_{r_1} f(t, r_1, r_2, \ldots, r_n) \tag{7}$$

Appling the dual property of the Fourier transform pair to the relationship shown in *PROPOSITION I* we have another pair which can be written as

$$\begin{aligned} H_n(t) &= A\Big\{\sum_{-N}^{N} C_r \exp(-j2\pi t r T_1)\Big\}\,\Pi(T_2 t) \\ \Leftrightarrow\ h_n(-f) &= \frac{A}{T_2}\Big[\sum_{-N}^{N} \mathbb{C}\Big]^n \mathrm{sinc}\left[\frac{-f}{T_2} - \frac{T_1}{T_2}\sum_{s=1}^{n} r_s\right] \\ &\text{for } n = 1, 2, 3, \ldots \end{aligned} \tag{8}$$

If we set the parameters as $T_1 = 1/T_0$ and $T_2 = 1/2\tau$ in (8), a useful formula appears. We call this the Representative Formula, which can be expressed as follows :

$$P_n(t) = A\{\sum_{r=-N}^{N} C_r \exp(-jr\omega_0 t)\}^n \Pi\left[\frac{t}{2\tau}\right], \ \omega_0 = \frac{2\pi}{T_0}$$

$$\Leftrightarrow \ P_n(f) = 2A\tau\left[\sum_{r=-N}^{N} C_r\right]^n \text{sinc}\left[2\tau f + \frac{2\tau}{T_0}\sum_{s=1}^{n} r_s\right]$$

$$\text{for } n=1,2,3,\ldots \tag{9}$$

Although this formula does not contain the case of n=0, defining the notations as

$$\left[\sum_{r=-N}^{N} C_r\right]^0 \equiv 1 \quad \text{and} \quad \sum_{s=1}^{0} r_s \equiv 0$$

we see that the formula proves to be right for n=0, and we obtain the well known pair in this case.

$$p_0(t) = A\,\Pi\left[\frac{t}{2\tau}\right] \Leftrightarrow P_0(f) = 2A\tau \text{ sinc } (2\tau f) \tag{10}$$

Comparing $p_n(t)$ in (9) with the problem function (1), we discover that the periodic function v(t) should be expressed as a truncated Fourier series in order to be applied to the model shown in Fig.1. On the other hand, it is well known that every periodic function can be accurately represented by the Fourier series. For comparison, we write the Fourier series of the function v(t) and its truncated version as follows :

$$v(t) = \sum_{r=-\infty}^{\infty} C_r \exp(-jr\omega_0 t) \text{ and}$$

$$v_N(t) = \sum_{r=-N}^{N} C_r \exp(-jr\omega_0 t), \ \omega_0 = \frac{2\pi}{T_0} \tag{11}$$

The two functions v(t) and $v_N(t)$ described above have the same form with the exception of the range of r. The truncated Fourier series $v_N(t)$ converges to the function v(t) when the parameter N approachs to infinity, provided that v(t) is sufficiently well behaved [6]. But the fact that N approachs to infinity means infinite number of taps in the model shown in Fig.1, and it is hard to deal with the Fourier analyses of such pulses using this model. We have to be given finite tabs or equivalently finite N so that the model can be recursively used to derive the Fourier transform of the pulse following its order. For finite N, a sufficient condition for $v(t) = v_N(t)$ can be given as follows [6] :

$$T_0\, v(t) = \int_{-T_0/2}^{T_0/2} v(z)\, S_N(z-t)\, dz \tag{12}$$

where

$$S_N(z-t) = \frac{\sin\{\left[N+\frac{1}{2}\right]\omega_0\,(z-t)\}}{\sin\{\frac{\omega_0}{2}\,(z-t)\}} \tag{13}$$

In a strict sense, every periodic function containing the pulse of interest as a part of itself must be tested whether or not satisfies the condition given in (12) prior to be applied to the method presented here. However, since it is obvious that the periodic functions obtained from the model by changing the tab coefficients and/or the number of tabs of the transversal filter satisfy the condition (12), we need not to carry out the test for such funcitions. Moreover, most of the pulses that frequently appear in the area of communication systems can be easily constructed from the model containing a few tabs. According to our experiences, many useful pulses could be obtained from the model composed of less than 5 tabs, that is N=1 or 2.

To summarize our interpretations discussed above, we can explain as follows : The Representative Formula given in (9) can be directly applied to all the pulses obtained by truncation of the periodic funcitions having finite N, and endow us with exact solutions. On the other hand, we can only approximate the solutions for pulses obtained from other periodic functions having infinite N. In the next section, we will only concentrate on the functions which can be expressed by finite Fourier coefficients, and find a more useful formula for deriving the Fourier transforms of them.

Ⅳ. FRAME FORMULA AND LOOK-UP TABLES

The largest trouble encountered when we apply the Representative Formula (9) to derive the Fourier transforms of pulses may be the complexity in expansion and calculation of multiple sunmations given by the definition (7). Even though the value of N is relatively small, a painstaking calculation must be carried out as the order of the pulse n increases. For the sake of overcoming this difficulty, we wish to present a more useful formula which contains only one summation in itself and all other parameters given by look-up tables.

When N is finite or the periodic funcition v(t) satisfies the condition (12), we can reorganize the Representative Formula (9) to submit new one, namely Generalized Frame formula, as the following proposition:

<*PROPOSTION 2* : Generalized Frame Formula>

Q(n) : The function $P_n(f)$ given in (9) can be reorganized resulting in the following form for all $n=1, 2, 3, \ldots$

$$P_n(f) = 2A\tau \sum_{i=0}^{t_n} b_{n,i} \operatorname{sinc}(2\tau f + d_{n,i}) \tag{14}$$

where $t_n = 2nN$

$$b_{n,i} = \sum_{x=0}^{2N} C_{N-X}\, b_{n-1,\, i-(2N-X)}$$

with assumptions

$$\begin{cases} b_{0,0} \equiv 1 \\ b_{p,q} \equiv 0, \text{ when } q<0 \text{ or } q>2pN \end{cases}$$

$$d_{n,i} = \frac{2\tau}{T_0}(i-nN), \text{ for } i=0, 1, 2, \ldots, t_n \qquad \blacksquare$$

We can obtain this formula as a result of a careful expansion of the multiple summations involved in the Representative Formula (9) using the definition (7). Detailed derivation of the Generalized Frame Formula (14) will be discussed in APPENDIX B.

It becomes clear that the Fourier transforms of pulse signals can be expressed by a single summation of shifted sinc(·) funcitions. About the other factors needed in the formula, we can make the following statements: The range of the summation t_n is easily decided by the order of pulse n and the limit of Fourier coefficients N. The coefficient of the sinc(·) funcition $b_{n,i}$ can be recursively obtained by the previous order nature of it starting from the initial poirt $b_{0,0}=1$. In addition, the delay factor of the sinc(·) function $d_{n,i}$ can be also given by the iterative equation in the formula. Moreover, once the form of a pulse or equivalently the periodic function v(t) is given, all these factors can be easily tabulated to form a look-up table by simple arithmetics related to the order of the pulse n.

Ultimately, we can draw an interesting conclusion. Tedious calculations or painstaking procedures are no longer needed to obtain the Fourier transforms of pulses like (1). As an alternative and handy method, we must propose a table look-up procedure with the Frame Formula given systematically by the characteristics of the periodic function corresponding to the pulse.

To clarify the procedure described above, two typical examples will be given in the next stage. Because the pulses involved here are frequently used in the area of communication systems, we sure that they are good references. We also hope the general tendency of this method can be understood by the readers in the process of carrying out the examples. Furthermore, one has to point out that the method can be applied to any other pulse which satisfies the condition disscussed at the end of Section Ⅲ.

As the primary example, we will investigate the Cosine Pulse which has been already discussed in our recent paper [3]. This yields the corresponding periodic funcition v(t) as

$$v(t) = \cos \frac{\pi t}{2\tau} \quad (15)$$

which has the fundamental period $T_0 = 4\tau$ and can be written in an exponential form

$$v(t) = \frac{1}{2} \{\exp[(j\omega_0 t) + \exp(-j\omega_0 t)\} \quad (16)$$

where the fundamental frequency is given by $\omega_0 = 2\pi / T_0$. Comparing (16) with (11), we see that the necessary factors appear as follows: $N=1$, $C_{-1} = C_1 = 1/2$, and other Fourier coefficients are all zero. Therefore, the parameters needed in the Frame Formula (14) are given by

$t_n = 2_n$

$b_{n,i} = (1/2)\ (b_{n-1,i-2} + b_{n-1,i})$

with assumptions

$$\begin{cases} b_{0,0} \equiv 1 \\ b_{p,q} \equiv 0, \text{ when } q<0 \text{ or } q>2p \end{cases}$$

$d_{n,i} = (1/2)\ (i-n)$, for $i = 0, 1, 2, \ldots, 2n$.

As stated above, these parameters can be shown as a look-up table. We submit TABLE I for this case. It must be noted that the medium range of TABLE I is the same as what we have constructed in |3| in spite of their different structures.

As another useful example, we deal with the Fourier transform of the n-th order Raised-Cosine Pulse, which has not been reported in any literature so far upto the best of the authors' knowledge. In this case, we are given the corresponding periodic function v(t) as follows:

$$v(t) = 1 + \cos\frac{\pi t}{\tau} = 1 + \frac{1}{2}\{\exp(-j\omega_0 t) + \exp(-j\omega_0 t)\} \quad (17)$$

Thus we have $T_0 = 2\tau$, $N=1$, $C_0 = 1$, $C_{-1} = C_1 = 1/2$, and other Fourier coefficients are all zero. Using again the relationships given in (14), the parameters are written in the forms

$t_n = 2n$

$b_{n,i} = (1/2)\ b_{n-1,i-2} + (1/2)\ b_{n-1,i}$

with assumptions

$$\begin{cases} b_{0,0} \equiv 1 \\ b_{p,q} \equiv 0, \text{ when } q<0 \text{ or } q>2p \end{cases}$$

$d_{n,i} = i - n$, for $i = 0, 1, 2, \ldots, 2n$.

As a result of almost the same process performed in the preceding example, we have TABLE II.

Both examples discussed above are considered as the extreme cases in a sense that the limit of subscripts N is commonly equal to 1. As another point of view, we can also present look-up tables according to change of N. We shall investigate the case of N=3 in order to confirm this fact and to expect the cases of higher N. First of all, the periodic function v(t) should have an accurate Fourier series truncated at the value N=3 so that the function may satisfy the condition (12). Thus we have

$$v(t) = \sum_{r=-3}^{3} C_r \exp(-jr\omega_0 t) \quad (18)$$

which directly shows us the necessary factors for the Frame Formula (14) in the following manner.

$t_n = 6n$

$b_{n,i} = C_3\ b_{n-1,i-6} + C_2\ b_{n-1,i-5} + C_1\ b_{n-1,i-4}$
$+ C_0\ b_{n-1,i-3} + C_{-1}\ b_{n-1,i-2} + C_{-2}\ b_{n-1,i-1} + C_{-3}\ b_{n-1,i}$

with assumptions

$$\begin{cases} b_{0,0} \equiv 1 \\ b_{p,q} \equiv 0, \text{ when } q<0 \text{ or } q>6p \end{cases}$$

$d_{n,i} = \frac{2\tau}{T_0}\ (i - 3n)$, for $i = 0, 1, 2, \ldots, 6n$

Based on these equations, we can build up the look-up table for N=3 as shown in TABLE III.

All the tables presented so far can be easily extended to any order by the simple arithmetics given in the lower part of each table. We have also offered the Fourier transform pair based on the Frame Formula in the first part of each table so that the look-up table can be self-contained and referenced quickly. The sketches sit on the upper right sides of TABLE I and II show the variation of pulse shape according to their order n, in time-domain. In both cases, we see that the pulse-width will be narrower and narrower as the order n increases.

Now, it must be noted that we can fabricate the look-up table for any other pulse in a similar manner expressed in this section, only if the corresponding periodic function is expressed, exactly or approximately, in a truncated Fourier series shown in (9). In that case, all the parameters needed to obtain the Fourier transform of the problem function (1) of any order can be located on the corresponding row of the table. Substituting the parameters at the appropriate positions in the Frame Formula, adequate result must be easily derived.

For example, given the 4-th order nature of the Raised-Cosine Pulse, all the necessary parameters can be located on the 4-th row of the medium range in TABLE II, resulting in

$$
\begin{aligned}
P_4(f) = \frac{2A\tau}{2^4}\{ & \mathrm{sinc}(2\tau f-4) + 8\,\mathrm{sinc}(2\tau f-3) \\
& + 28\,\mathrm{sinc}(2\tau f-2) \\
& + 56\,\mathrm{sinc}(2\tau f-1) + 70\,\mathrm{sinc}(2\tau f) \\
& + 56\,\mathrm{sinc}(2\tau f+1) \\
& + 28\,\mathrm{sinc}(2\tau f+2) + 8\,\mathrm{sinc}(2\tau f+3) \\
& + \mathrm{sinc}(2\tau f+4)\}
\end{aligned}
\tag{19}
$$

V. CONCLUDING REMARKS

In this paper we generalized the method which had been presented in our recent paper [3]. We wanted to describe an easy and compact method for deriving the Fourier transforms of pulse signals throughout the papers. Consequently, the Fourier transforms of all pulses that satisfies the condition (12) could be easily obtained through the substituting processes using the corresponding look-up tables and the Frame Formulas, no matter what orders of them would be given. Although the look-up tables for many other useful pulses were not offered here, we had already prepared them for references. The Cosine-Pulse and the Raised-Cosine Pulse shown in the last section are the typical cases of them. As stated above, it must be noted that the width of pulse measured at the FWHM(full width half maximum) has an apparent tendency to be narrower as the order of the pulse increases. We expect this tendency will play an importent role in analysis of the characteristics of ultra short pulse transmission. This is a subject of our future work.

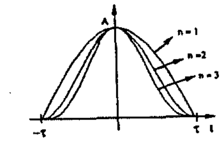
On the other hand, some numerical methods for deriving the Fourier transforms of higher order pulses can be presented using conventional algorithms such as Cooley-Tukey[7]. Moreover, our look-up tables will be used to calculate the function values for each sampling instants just by referencing the memory which contains the contents of the look-up tables. It is expected that this method can reduce the transformation time for useful higher order pulse signals. This is another object of our future works.

TABLE Ⅰ Look-up Table for the Fourier Transforms of the n-th Ordeor Cosine Pulses

$$p_n(t) = A\left[\cos\frac{\pi t}{2\tau}\right]^n \Pi\left[\frac{t}{2\tau}\right]$$

$$\Leftrightarrow P_n(t) = 2A\tau\sum_{i=0}^{2n} M_n \cdot B_{n,i}\ \mathrm{sinc}\left[2\tau f + \frac{1}{2}D_{n,i}\right]$$

where $b_{n,i} = M_n \cdot B_{n,i}$ and $d_{n,i} = (1/2)D_{n,i}$



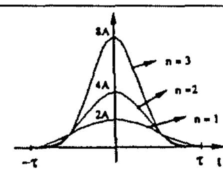
n	M_n	$B_{n,i}$	$D_{n,i}$
0	1	1	0
1	1/2	1 0 1	-1 0 1
2	$1/2^2$	1 0 2 0 1	-2 -1 0 1 2
3	$1/2^3$	1 0 3 0 3 0 1	-3 -2 -1 0 1 2 3
4	$1/2^4$	1 0 4 0 6 0 4 0 1	-4 -3 -2 -1 0 1 2 2 4
5	$1/2^5$	1 0 5 0 10 0 10 0 5 0 1	-5 -4 -3 -2 -1 0 1 2 3 4 5
6	$1/2^6$	1 0 6 0 15 0 20 0 15 0 6 0 1	-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6
○	○	○	○
○	○	○	○
○	○	○	○
Equations for extension of the TABLE	$B_{0,0} \equiv 1$ $B_{p,q} \equiv 0$, if $q<0$ or $q>2p$ $B_{n,i} = B_{n-1,i-2} + B_{n-1,i}$ for $n = 1, 2, 3, \ldots$ $i = 0, 1, 2, \ldots, 2n$	$D_{0,0} \equiv 0$ $D_{n,i} = i - n$ for $n = 1, 2, 3, \ldots$ $i = 0, 1, 2, \ldots, 2n$	

TABLE Ⅱ Look-up Table for the Fourier Transforms of the n-th Order Raised-Cosine Pulses

$$p_n(t) = A\left[1+\cos\frac{\pi t}{\tau}\right]^n \Pi\left[\frac{t}{2\tau}\right]$$

$$\Leftrightarrow P_n(t) = 2A\tau\sum_{i=0}^{2n} M_n \cdot B_{n,i}\ \mathrm{sinc}\left[2\tau f + d_{n,i}\right]$$

where $b_{n,i} = M_n \cdot B_{n,i}$



n	M_n	$B_{n,i}$	$D_{n,i}$
0	1	1	0
1	1/2	1 2 1	-1 0 1
2	$1/2^2$	1 4 6 4 1	-2 -1 0 1 2
3	$1/2^3$	1 6 15 20 15 6 1	-3 -2 -1 0 1 2 3
4	$1/2^4$	1 8 28 56 70 56 28 8 1	-4 -3 -2 -1 0 1 2 3 4
5	$1/2^5$	1 10 45 ⊔ 210 ⊔ 210 ⊔ 45 10 1	-5 -4 -3 -2 -1 0 1 2 3 4 5
		120 252 120	
○	○	○	○
○	○	○	○
○	○	○	○

Equations for extension of the TABLE II	$B_{0.0} \equiv 1$ $B_{p.q} \equiv 0$, if $q<0$ or $q>2p$ $B_{n.i} = B_{n-1,i-2} + 2B_{n-1,i-1} + B_{n-1,i}$ for $n=1,2,3,\ldots$ $i=0,1,2,\ldots,2n$	$d_{0,0} \equiv 1$ $d_{n,i} = i-n$ for $n=1,2,3,\ldots$ $i=0,1,2,\ldots,2n$

TABLE III Look-up Table for the Fourier Transforms of Pulses with N=3

$$p_n(t) = A\left\{\sum_{r=-3}^{3} C_r \exp(-jr\omega_0 t)\right\}^n \Pi\left[\frac{t}{2\tau}\right] \Leftrightarrow P_n(f) = 2A\tau\sum_{i=0}^{6n} b_{n,i}\,\mathrm{sinc}\left[2\tau f + d_{n,i}\right]$$

n	$b_{n,i}$	$d_{n,i}$
0	$b_{00}=1$	$d_{00}=0$
1	$b_{10}\ b_{11}\ b_{12}\ b_{13}\ b_{14}\ b_{15}\ b_{16}$	$d_{10}\ d_{11}\ d_{12}\ d_{13}\ d_{14}\ d_{15}\ d_{16}$
2	$b_{20}\cdots b_{25}\ b_{26}\ b_{27}\cdots b_{2,12}$	$d_{20}\ d_{22}\ d_{22}\ d_{23}\cdots d_{2,11}\ d_{2,12}$
3	$b_{30}\cdots b_{38}\ b_{39}\ b_{3,10}\cdots b_{3,18}$	$d_{30}\ d_{31}\ d_{32}\ d_{33}\cdots d_{3,17}\ d_{3,18}$
o	o	o
o	o	o
o	o	o
E	$b_{00} \equiv 1$	
Q	$b_{pq} \equiv 0$, when $q<0$ or $p>6_p$	$d_{00} \equiv 0$
A	$b_{n,i} \equiv C_3\, b_{n-1,i-6} + C_2\, b_{n-1,i-5}$	
T	$+ C_1\, b_{n-1,i-4} + C_0\, b_{n-1,i-3}$	$d_{n,i} = 2\tau(i-3n)/T_0$
I	$+ C_{-1}\, b_{n-1,i-2} + C_{-2}\, b_{n-1,i-1}$	for $n=1,2,3,\ldots$
O	$+ C_{-3}\, b_{n-1,i}$	$i=0,1,2,\ldots,6_n$
N	for $n=1,2,3,\ldots$	
S	$i=0,1,2,\ldots,6_n$	

APPENDIX A
PROOF OF PROPOSITION 1

First, we see that the proof involved in the function $H_n(f)$ for $n=1,2,3,\ldots$ is trivial by induction using the relations (2) and (6). Hence we only concerned with the proof on $h_n(t)$ for all n. This can be also proved by the principle of mathematical induction in the following manner.

Basis step : When $n=1$, we obtain the following relationship using (6) with the initial funcition (3) :

$$h_1(t) = k(t) * h_0(t) = \sum_{r=-N}^{N} C_r\, h_0(t - rT_1)$$
$$= \frac{A}{T_2} \sum_{r=-N}^{N} C_r\, \mathrm{sinc}\left[\frac{t}{T_2}\ \frac{T_1}{T_2}\ r\right] \qquad (A1)$$

which indicates that statement $P(1)$ is true.

Induction step : Suppose that statement $P(m)$ is true for some integer $m \geq 1$, then we have

$$h_m(t) = \frac{A}{T_2}\left[\sum_{N}^{N}\ \right]^m \mathrm{sinc}\left[\frac{t}{T_2}\ \frac{T_1}{T_2}\sum_{s=1}^{m} r_s\right] \qquad (A2)$$

After one additional step applying (3), we obtain the following expression for the (m+1)-st order :

$$h_{m+1}(t)=k(t)*h_m(t)=\sum_{r=-N}^{N} C_r\, h_m(t-rT_1)$$

$$=\sum_{r_{m+1}=-N}^{N} C_{r(m+1)}\left\{\frac{A}{T_2}\left[\sum_{=-N}^{N} \copyright\right]^m \operatorname{sinc}\left[\frac{t}{T_2}-\frac{T_1}{T_2}\, r_{m+1}-\frac{T_1}{T_2}\sum_{s=1}^{m} r_s\right]\right\}$$

$$=\left\{\frac{A}{T_2}\left[\sum_{=-N}^{N} \copyright\right]^{m+1} \operatorname{sinc}\left[\frac{t}{T_2}-\frac{T_1}{T_2}\sum_{s=1}^{m+1} r_s\right]\right\} \qquad \text{(A3)}$$

In this expression, the definition (7) has been used again in a recursive manner. The result proves that statement $P(m+1)$ is also true. Therfore, the statement $P(n)$ is true for all $n=1, 2, 3, \ldots$ by the principle of mathematical induction.

APPENDIX B
PROOF OF PROPOSITION 2

Basis step: When $n=1$, we have following equations from the Represetative Formula given in (9) and the definition (7)

$$P_1(f)=2A\tau\sum_{r_1=-N}^{N} C_{r_1} \operatorname{sinc}\left[2\tau f+\frac{2\tau}{T_0}\, r_1\right] \qquad \text{(B1)}$$

After making the change of variable $r_1=i-N$ and substituting, we can write

$$P_1(f)=2A\tau\sum_{r_1=-N}^{t_1} b_{1,i} \operatorname{sinc}\,(2\tau f+d_{1,i}) \qquad \text{(B2)}$$

where $t_1=2N$

$$b_{1,i}=C_{i-N}=\sum_{x=0}^{2N} C_{N-x}\, b_{0,i-(2N-x)}$$

with the same assumptions in *PROPOSITION 2*

$$d_{1,i}=\frac{2\tau}{T_0}\,(i-N),\ \text{for } i=0,1,2,\ldots,2N$$

which indicates that satement $Q(1)$ is true.

Induction step: If we assume that statement $Q(k)$ is true for some integer $k\geq 1$, then we obtain the following relationships :

$$P_k(f)=2A\tau\left[\sum_{=-n}^{N} \copyright\right]^k \operatorname{sinc}\left[2\tau f+\frac{2\tau}{T_0}\sum_{s=1}^{k} r_s\right]$$

$$=2A\tau\sum_{i=0}^{t_k} b_{k,i} \operatorname{sinc}\,(2\tau f+d_{k,i}) \qquad \text{(B3)}$$

where

$$t_k=2kN$$

$$b_{k,i}=\sum_{x=0}^{2N} C_{N-x}\, b_{k-1,i-(2N-x)}$$

with the same assumptions in *PROPOSITION 2*

$$d_{k,i}=\frac{2\tau}{T_0}\,(i-kN),\ \text{for } i=0,1,2,\ldots,2kN$$

The Representative Formula (9) for the $(k+1)$-st order can be written as

$$P_{k+1}(f)=2A\tau\left[\sum_{=-N}^{N} \copyright\right]^{k+1} \operatorname{sinc}\left[2\tau f+\frac{2\tau}{T_0}\sum_{s=1}^{k+1} r_s\right] \qquad \text{(B4)}$$

After applying the recursive property of the definition (7) and substituting the shifted version of (B3), equation (B4) can be expressed in the form

$$P_{k+1}(f)=2A\tau\sum_{r_{k+1}=-N}^{N} C_{r_{k+1}}\left[\sum_{=-N}^{N} \copyright\right]^k \operatorname{sinc}\left\{2\tau\left[f+\frac{r_{k+1}}{T_0}\right]+\frac{2\tau}{T_0}\sum_{s=1}^{k} r_s\right\}$$

$$=\sum_{r_{k+1}=-N}^{N} C_{r_{k+1}}\, P_k\left[f+\frac{r_{k+1}}{T_0}\right]$$

$$=2A\tau\sum_{r_{k+1}=-N}^{N} C_{r_{k+1}}\left\{\sum_{i=0}^{t_k} b_{k,i} \operatorname{sinc}\left[2\tau f+\frac{2\tau}{T_0}\, r_{k+1}+d_{k,i}\right]\right\} \qquad \text{(B5)}$$

where $t_k, b_{k,i}$, and $d_{k,i}$ have the same values are that given in (B3). Exchanging the order of the summations in (B5) and substituting the values of t_k and $d_{k,i}$, we have

$$P_{k+1}(f)$$

$$=2A\tau\sum_{i=0}^{2kN} b_{k,i}\left[\sum_{r_{k+1}=-N}^{N} C_{r_{k,i}} \operatorname{sinc}\left\{2\tau f\ \frac{2\tau}{T_0}\,(i-kN+r_{k+1})\right\}\right] \qquad \text{(B6)}$$

In the next stage, we have to expand the summations in the above expression about the somewhat complex and large number of terms, we can combine them according to their delay factors of sine(·) funcitions, and the resulting equation becomes
delay factors of sine(·) funcitions, and the resulting equation becomes

$$P_{k+1}(f) = 2A\tau \sum_{i=0}^{t_{k+1}} b_{k+1,i} \, \mathrm{sinc} \, (2\tau f + d_{k+1,i}) \qquad (B7)$$

where

$$t_{k+1} = 2(k+1)N$$

$$b_{k+1,i} = \sum_{x=0}^{2N} C_{N-x} \, b_{k,i-(2N-x)}$$

with the same assumptions in *PROPOSITION 2*

$$d_{k+1,i} = \frac{2\tau}{T_0}\{i-(k+1)N\}, \text{ for } i=0,1,2,\ldots,2(k+1)N$$

These notify that statement $Q(k+1)$ is also true. Hence, the statement $Q(n)$ is true for all n = 1, 2, 3, ... by the principle of mathematical induction.

본 논문은 1992년도 통신학술단체 육성지원금에 의하여 이루어졌습니다.

REFERENCES

1. J. W. Cooley, P. A. Lewis, P. D. Welch, "The fast Fourier transform and its applications," IEEE Trans. Education, Vol.E-12, pp.27-34, Mar. 1969.
2. W. Zhao, E. Bourkoff, "Femtosecond pulse propagation in optical fibers : Higher order effects," IEEE Jour. of Quantum Electron., Vol. 24, No.2, pp.365-372. Feb. 1988.
3. Chang E. Kang, Yong S. OH, "The look-up table for deriving the Fourier transforms of cosine-pulses," to be published in IEEE Trans. Education, in 1993.
4. P. Kabal, S. Pasupathy, "Partial-response signaling," IEEE Trans. Commun., Vol. Com-23, pp.921-934, Sep. 1975.
5. A. B. Carlson, *Communication systems-An introduction to signals and noise in electrical communication*, 3rd ed., Prentice-Hall, pp.60-64, 1986.
6. H. Stark, F. B. Tuteur, *Modern electrical communications-Theory and systems*, Prentice-Hall, pp.27-29, 1979.
7. C. S. Burrus, T. W. Parks, *DFT/FFT and convolution algorithms-Theory and implementation*, edited by Texas Instruments, Inc. and Published by John Wiley and Sons 1985.

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