Category of H-fuzzy Semitopogenous Spaces

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ABSTRACT

In this paper, we introduce the notion of H-fuzzy semitopogenous spaces. In section 1, we give the preliminary definitions and some basic results. In section 2, we show that category HFS of H-fuzzy semitopogenous spaces and continuous maps between them is topological and cotopological. Using ordinary operations, we characterize coreflective subcategories and then show that each of Top, Prox, Qunif, and Unif is isomorphic with some coreflective subcategory of HFS. Moreover, we show that sa-HFS is closed under the formation of initial sources in a-HFS, whewe a is a symmetrical elementary operation.

I. Preliminaries.

Throughout this paper, we will let H denote the complete Heyting algebra (H, $\vee$, $\wedge$, $*$) with order reversing involution $*$. 0 and 1 denote the supremum and the infimum of ϕ, respectively. Given a set X, any element of H^X is called H-fuzzy set (or, simply f. set) in X and will be denoted by small Greek letters, such as μ, ν, ρ, σ. H^X inherites a structure of H with order reversing involution in natual way, by defining $\vee$, $\wedge$, $*$ pointwise (same notations of H are usual).

If f is a map from a set X to a set Y and $\mu\in H^Y$, then $f^{-1}(\mu)$ is the f. set in X defined by $f^{-1}(\mu)(x)=\mu(f(x))$. Also for $\sigma\in H^X$, $f(\sigma)$ is the f. set in Y defined by $f(\sigma)(y)=\sup\{\sigma(x): f(x)=y\}$ ([3]).

A relation $\sqsubset$ on H^X is called a H-fuzzy semitopogenous (or, simply, fs.) order on X if it satisfies the following axioms :

SO1) $0 \sqsubset 0$ and $1 \sqsubset 1$,

SO2) $\mu \sqsubset \rho$ implies $\mu \leq \rho$.

SO3) $\mu_1 \leq \mu \sqsubset \rho \leq \rho_1$ implies $\mu_1 \sqsubset \rho_1$.

Let $\ll_1$ and $\ll_2$ be fs. orders on a set X. The composition $\ll = \ll_1 \circ \ll_2$ is defined by $\mu \ll \rho$ iff there exists f. set σ in X such that $\mu \ll_2 \sigma \ll_1 \rho$. It is easy to see that $\ll$ is a fs. order on X. For a fs. order $\ll$, we will usually write $\ll^2$ for composition $\ll \circ \ll$.

The complement of a fs. order $\sqsubset$ is the fs. order $\sqsubset^c$ which is defined by $\mu \sqsubset^c \rho$ iff $\rho^* \sqsubset \mu^*$. It is easy to show that if$\{\sqsubset_i : i\in I\}$ is a family of fs. orders in a set X, then $(\cup\{\sqsubset_i : i\in I\})^c = \cup\{\sqsubset_i^c : i\in I\}$.

A fs. order $\sqsubset$ is called :

1) symmetrical if $\sqsubset = \sqsubset^c$.

2) topogenous if $\mu_1 \sqsubset \rho_1$ and $\mu_2 \sqsubset \rho_2$ imply $\mu_1 \vee \mu_2 \sqsubset \rho_1 \vee \rho_2$ and $\mu_1 \wedge \mu_2 \sqsubset \rho_1 \wedge \rho_2$.

3) perfect if $\mu_i \sqsubset \rho_i$, $i\in I$, implies $\sup \mu_i \sqsubset \sup \rho_i$.

4) biperfect if $\mu_i \sqsubset \rho_i$, $i\in I$, implies $\sup \mu_i \sqsubset \sup \rho_i$ and $\inf \mu_i \inf \rho_i$.

A fs. order is called finer than another one $\lll$ if $\mu \lll \rho$ implies $\mu \sqsubset \rho$. In this case we also say that $\lll$ is coarser than $\sqsubset$.

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The following propositions are easily established :

Proposition 1.1 Let be a fs. order on a set X. Then one has the following :

1) there exists a topogenous fs. order $\sqsubset^q$ finer than $\sqsubset$ and coarser than any topogenous fs. order on X which is finer than $\sqsubset$. It is defined by $\mu \sqsubset^q \rho$ iff there are natural numbers m,n and μ_i, i=1, 2, ..., m and ρ_j, j=1, 2, ..., n, such that $\mu = \vee \mu_i$, $\rho = \wedge \rho_j$ and $\mu_i \sqsubset \rho_j$ for all i=1, 2, ..., m and j=1, 2, ..., n.

2) there exists a perfect fs. order $\sqsubset^q$ finer than $\sqsubset$ and coarser than any perfect fs. order on X which is finer than $\sqsubset$. It is defined by $\mu \sqsubset^q \rho$ iff there is a family $\{\mu_i : i \in I\}$ of f. sets such that $\mu = \sup \mu_i$ and $\mu_i \sqsubset \rho$ for all $i \in I$.

3) there exists a biperfect fs. order $\sqsubset^b$ finer than $\sqsubset$ and coarser than any biperfect fs. order on X which is finer than $\sqsubset$. It is defined by $\mu \sqsubset^b \rho$ iff there are families $\{\mu_i : i \in I\}$, $\{\rho_j : j \in J\}$ of f. sets such that $\mu = \sup \mu_i$, $\rho = \inf \rho_j$ and for all $i \in I$ and all $j \in J$.

4) $\sqsubset = (\sqsubset \cup \sqsubset^c)$ is symmetrical fs. order finer than $\sqsubset$ and coarser than any symmetrical fs. order on X which is finer than $\sqsubset$.

Proposition 1. 2 Let f be a map from a set X to a set Y and let $\sqsubset$ and $\ll$ be fs. orders on X and Y, respectivly. Then one has the following :

(1) let $\sqsubset$ be a fs. order on Y. Define a relation $\ll$ on X by $\mu \ll \rho$ iff $f(\mu) \sqsubset (f(\rho^*))^*$. Then $\ll$ is a fs. order on X. we will call $\ll$ the inverse image of $\sqsubset$ by the map f and we will denote it by $f^{-1}(\sqsubset)$.

(2) let $\sqsubset$ be a fs. order on X. Define a relation $\ll$ on Y by $\mu \ll \rho$ iff $\mu \le \rho$ and $f^{-1}(\mu) \sqsubset f^{-1}(\rho)$. Then $\ll$ is a fs. order on Y. We will call $\ll$ the image of $\sqsubset$ by the map f and we will denote it by $f(\sqsubset)$.

(3) $\mu \ll \rho$ implies $f^{-1}(\mu)\ f^{-1}(\mu \ll \rho)\ f^{-1}(\rho)$.

(4) for μ, ρ in X, $\mu\ f^{-1}(\ll)\ \rho$ iff there are μ_1, ρ_1 in Y such that $\mu_1 \ll \rho_1$, $\mu \le f^{-1}(\mu_1)$ and $f^{-1}(\rho_1) \le \rho$.

(5) if f is onto, then $\mu f(\sqsubset)\ \rho$ iff there are μ_1, ρ_1 in X such that$f^{-1}(\mu) \le \mu_1 \sqsubset \rho_1 \le f^{-1}(\rho)$.

(6) $\ll$ is coarser than $f(f^{-1}(\ll))$.

(7) $\sqsubset$ is finer than $f^{-1}(f(\sqsubset))$.

(8) if $\ll_1$ is a fs. order on Y which is finer than $\ll$ then $f^{-1}(\ll_1)$ is finer than $f^{-1}(\ll)$.

(9) if $\sqsubset_1$ is a fs. order on X which is finer than $\sqsubset$ then $f(\sqsubset_1)$ is finer than $f(\sqsubset)$.

(10) $f(\sqsubset)$ is the finest fs. order $\sqsubset_1$ on Y for which $\mu \sqsubset_1 \rho$ in Y implies that $f^{-1}(\mu) \sqsubset f^{-1}(\rho)$.

(11) $f^{-1}(\ll)$ is the coarsest fs. order $\ll_1$ on X for which $\mu \ll \rho$ implies that $f^{-1}(\mu) \ll_1 f^{-1}(\rho)$.

(12) if $\{\sqsubset_i : i \in I\}$ is a family of fs. orders on Y, then $f^{-1}(\cup \sqsubset_i) = \cup f^{-1}(\sqsubset_i)$ and $f^{-1}(\cap \sqsubset_i) = \cap f^{-1}(\sqsubset_i)$.

(13) if $\{\ll_i : i \in I\}$ is a family of fs. order on X, then $f(\cup \ll_i) = \cup f(\ll_i)$ and $f(\cap \ll_i) = \cap f(\ll_i)$.

(14) if g is a map from a set Y to a set Z and $\ll_2$ is a fs. order on Z, then $(g \circ f)^{-1}(\ll_2) = f^{-1}(g^{-1}(\ll_2))$ and $(g \circ f)(\sqsubset) = g(f(\sqsubset))$.

(15) $f^{-1}(\ll^2) \subseteq f^{-1}(\ll)^2$.

(16) $f^{-1}(\ll^c) = (f^{-1}(\ll))^c$.

Definition 1. 4 Let SO(X) denote the set of all fs. orders on a set X. (1) A unary operation ^{a} on SO(X) will be called an elementary operation if it satisfies the following axioms :

E1) $\sqsubset$ is coarser than $\sqsubset^a$.

E2) $\sqsubset^{aa} = \sqsubset^a$

E3) $\sqsubset$ is coarser than $\ll$ implies $\sqsubset^a$ is coarser than $\ll^a$.

E4) $\sqsubset^{2a}$ is coarser than $\sqsubset^{a2}$.

E5) If f is a map from a set X to a set Y and $\ll$ is fs. order on Y, then $f^{-1}(\ll^a) = (f^{-1}(\ll))^a$.

(2) An elementary operation ^{a} is said to be symmetrical if $\sqsubset^{ac} = \sqsubset^{ca}$ for any fs. order $\sqsubset$ on X.

The following is immediate from the definition.

Proposition 1. 5 Let $\{\ll_i : i \in I\}$ be a family of fs. order on a set X and ^{a} an elementary operation. Then $(\cup\{\ll_i : i \in I\})^a = (\cup\{\ll_i^a : i \in I\})^a$.

Example 1) The identity operation ^{i} efined by $\sqsubset^i = \sqsubset$ is an elementary operation.

2) The operations ^{q}, ^{p}, ^{b}, ^{qp}, ^{sq}, ^{sb} are elementary oprations.

3) The operations ^{i}, ^{q}, ^{b} are symmetrical elementary operations.

The following proposition is easily established :

Proposition 1. 6 Let f be a map of a set X to a set Y. and $\sqsubset$ a fs. order on X. Then one has the following :.

1) if ^{a} is an elementary operartion , then $f(\sqsubset)^a \subseteq f(\sqsubset^a)$.

2) $f(\sqsubset^c) = (f(\sqsubset))^c$.

II. Category HFS and its subcategories.

We call an order family on X a nonempty subset of SO(X). An order family A is called finer than another one B ($B < A$) if for each $\sqsubset$ in B there is a fs. order $\ll$ in A finer than $\sqsubset$. In this case we also say that B is coarser than A. Order families A and B are said to be equivalent (denoted by $A \cong B$) if $A < B$ and $B < A$.

Definition 2. 1 An order family S on a set X is said to be H-fuzzy semitopogenous (or, simply hfs.) structure on X if it satisfies the following properties :

S1) S is directed in the sense that given any two members of S there exists a member of S finer than both.

S2) S is interpolated in the sense that for each in S there exists a fs. order in S such that is coarser than $\ll^2$.

The pair (X,S) is called a H-fuzzy semitopogenous (or, simply hfs.) space. Let S, T be hfs. structures on X, Y, respectively. A map f from X to Y is said to be continuous if $f^{-1}(T) < S$, where $f^{-1}(T) = \{f^{-1}(\ll) : \ll \in T\}$.

With HFS we will denote the category whose objects are hfs. spaces and whose morphisms are continuous maps.

Example 1) For any set X, $\{\leq\}$ is a clearly hfs. structure on X and (X, $\{\leq\}$) is a called the discrete hfs. space.

2) For any set X, a relation $\sqsubset_{0,1}$ on X defined by $\mu \sqsubset_{0,1} \rho$ iff $\mu = 0$ or $\rho = 1$, is a clearly hfs. structure on X and (X, $\{\sqsubset_{0,1}\}$) is called the indiscrete hfs. space.

The following is easily established :

Proposition 2. 2 Let A be an order family on a set X. Then one has the following :

1) there exists a directed order family A^g finer than A and coarser than any other directed order families finer than A. In fact, $A^g = \{\cup B : B$ is a nonempty finite subset of $A\}$.

2) there exists a interpolated order family A^2 coarser than A and finer than any other interpolated order families coarser than A. In fact, $A^2 = \{\sqsubset :$ there is a sequence $(\sqsubset_n)$ in $T(A)$ such that $\sqsubset = \sqsubset_1$ and for each n, $\sqsubset_n$ is coarser than $\sqsubset_{n+1}{}^2\}$, where $T(A) = \{\sqsubset : \sqsubset$ is a fs. order on X and there is a fs. order $\ll \in A$ which is finer than $\sqsubset\}$.

3) If A is a directed order family, then A^2 is a hfs. structure on X.

4) If A is an interpolated order family, then A^g is a hfs. structure on X.

5) A is a hfs. structure on X iff $A^g < A < A^2$.

Theorem 2. 3 HFS is a topological and cotopological category. In particular, (1) For a set X, a family $((X_i, S_i))_{i\in I}$ of HFS indexed by a class I, and a source $(f_i : X \to X_i)_{i\in I}$, let $S = (\cup f_i^{-1}(S_i))^g$. Then S is the initial hfs. structure on X with respect to $(f_i)_{i\in I}$.

(2) For a set X, a family $((X_i, S_i))_{i\in I}$ of HFS indexed by a class I, and a sink $(f_i : X_i : \to X)_{i\in I}$, let $S = \{\cap f(\sqsubset_i) : (\sqsubset_i) \in \Pi S_i\}$. Then S^2 is the final hfs. structure on X with respect to $(f_i)_{i\in I}$.

Proof. Let us show that HFS is a topological category. To do so, it is enough to show that (1) holds. It is follows from (1. 2. 10) and (2. 2. 4) that S is a hfs. structure on X. By the definition of S, it is clear that for each $i \in I$, $f_i : (X, S) \to (X_i, S_i)$ is continuous. Suppose $(Y, T) \in$ HFS, $g : Y \to X$ is a map and $f_i g : (Y, T) \to (X_i, S_i)$ is a continuous map for all $i \in I$. Let $\sqsubset \in S$. Then there exists a finite subset F of I such that $\sqsubset = \cup\{f_i^{-1}(\sqsubset_i) : i \in I\}$. By (1. 2. 12), $g^{-1}(\sqsubset) = \{g^{-1}(f_i^{-1}(\sqsubset_i) : i \in I\}$. Since fig is continuous, for each $i \in F$, there exists $\ll_i$ in T such that $g^{-1}(f_i^{-1}(\sqsubset_i))$ is coarser than $\ll_i$. Since T is a hfs. structure on Y, there exists $\ll$ in T finer than $\cup \ll_i$. Thus g is a continuous map. It is known [11] that every topological category is cotopological. However, for the further development, we prove the second statement. It follows from (1. 2. 9) and (2. 2. 3) that S^2 is hfs. structure on X. By the definition of S^2, it is obvious that each $i \in I$, $f_i : (X_i, S_i) \to (X, S^2)$ is continuous. Suppose (Y, T) HFS and $g : X \to Y$ is a map and $gf_i : (X_i, S_i) \to (Y, T)$ is a continuous map for all $i \in I$. Let $\ll \in T$. Since for each $i \in I$, gf_i is continuous, there exists $\sqsubset_i$ in S_i such that $f_i^{-1}g^{-1}(\ll)$ is coarser than $\sqsubset_i$. Then for each $i \in I$, $g^{-1}(\ll)$ is coarser than $f_i^{-1}(\sqsubset_i)$, and hence $g^{-1}(\ll)$ is coarser than $\cap f_i(\sqsubset_i)$. Thus g is continuous. This completes the proof.

Remark It is immediate from theorem in [11] and the above theorem that HFS is complete and cocomplete.

Definition 2. 4 (1) A unary operation ^{k} on SO(X) will be called an ordinary operation if it satisfies the following axioms :

O1) $A < A^k$.

O2) $A^{kk} = A^k$.

O3) If $A < B$ then $A^k < B^k$.

O4) $A^{2k} < A^{k2}$.

O5) $A^{kg} < A^{kg}$.

O6) If f is a map of a set X into Y and A is an order family on Y, then $f^{-1}(A^k) = (f^{-1}(A))^k$.

(2) Let ^{k} and ^{l} be ordinary operations such that $A^k < A^l$ for any order family A, then we say that k is coarser than l.

(3) Let ^{k} be a ordinary operation. Then a hfs. structure S on a set X is said to be ^{k}-hfs. structure if $S = S^k$.

Example 1. Let A be an order family on a set X. Then $t : A \rightarrow \cup A$ is an ordinary operation.

2. ^{g} is an ordinary operation.

3. every elementary operation is an ordinary operation.

4. if ^{a} is an elementary operation, then ^{ga} and ^{ta} are ordinary operation.

Notation Let ^{k} be an ordinary operation. Then ^{k}-HFS denotes the full subcategory of HFS determined by all ^{k}-hfs. spaces.

Theorem 2. 5 If ^{k} is coarser than ^{l} then ^{l}-HFS is coreflective in ^{k}-HFS.

Proof. Let $(X, S) \in {}^k$-HFS. It is clear that S^l is ^{l}-hfs. structure on X and the identity map $l_X : (X, S^l) \rightarrow (X, S)$ is continuous. Take any (Y, T) in ^{l}-HFS and any continuous map $f : (Y, T) \rightarrow (X, S)$. Then $f^{-1}(S^l) = (f^{-1}(S))^l < T^l$. Since $T^l = T$, $f : (Y, T) \rightarrow (X, S^l)$ is continuous. This completes the proof.

Corollary 2. 6 1) If ^{k} is an ordinary operation, then ^{k}-HFS is coreflective in HFS.

2) ^{qa}-HFS is coreflective in ^{q}-HFS, where ^{a} is an elementary operation.

3) ^{tqa}-HFS is coreflective in ^{tq}-HFS, where ^{a} is an elementary operation.

4) ^{bs}-HFS is coreflective in ^{b}-HFS.

Theorem 2. 7 The category TOP of H-fuzzy topological spaces (and continuous maps) and ^{tqp}-HFS are isomorphic.

[A H-fuzzy topology (or, simply topology) is a subset τ of H^X having the following :

T1) $0, 1 \in \tau$

T2) If $\mu, \rho \in \tau$ then $\mu \wedge \rho \in \tau$.

T3) $\mu_i \in \tau$, $i \in I$ implies $\vee \mu_i \in \tau$.

The pair (X, τ) is called a H-fuzzy topological (or, simply topological) space. The members of τ are then said to be a τ-open fuzzy set in X, or merely open set in X if no confusion may result. A map f of a topological space X into another one Y is called continuous if $f^{-1}(\mu)$ is open in X for each open set in Y.]

Proof. For any (X, τ) in TOP, define $\sqsubset_\tau$ as follows : $\mu \sqsubset_\tau \rho$ iff $\mu \leq \sigma \leq \rho$ for some $\sigma \in \tau$. It is clear that $(X, \sqsubset_\tau) \in {}^{tqp}$-HFS. If $f : (X, \tau) \rightarrow (Y, \tau_1)$ is continuous and suppose $\mu \sqsubset_{\tau_1} \rho$. By the definition of $\sqsubset_\tau$, there exists $\sigma \in \tau_1$ such that $\mu \leq \sigma \leq \rho$. Since $f^{-1}(\mu) \leq f^{-1}(\sigma) \leq f^{-1}(\rho)$ and $f \in$ TOP, $f^{-1}(\sigma) \in \tau$ and hence $f^{-1}(\mu) \sqsubset_\tau f^{-1}(\rho)$. Thus $F : \text{TOP} \rightarrow {}^{tqp}\text{-HFS}(F(X, \tau) = (X, \sqsubset_\tau)$ and $F(f) = f)$ is a functor. For any $(X, \sqsubset)$ in ^{tqp}-HFS, let $\tau_\sqsubset = \{\mu \in H^X : \mu \sqsubset \mu\}$. It is clear that $(X, \tau_\sqsubset) \in$ TOP. Suppose that $f : (X, \sqsubset) \rightarrow (Y, \ll) \in {}^{tqp}$-HFS and $\mu \in \tau_\ll$. Then $\mu \ll \mu$. Since $f \in {}^{tqp}$-HFS, $f^{-1}(\mu) \sqsubset f^{-1}(\mu)$ and hence $f^{-1}(\mu)$ is an open set in X. Thus $G : {}^{tcp}\text{-HFS} \rightarrow \text{TOP}$ $(G(X, \sqsubset) = (X, \tau_\sqsubset)$ and $F(f) = f)$ is a functor. For any $(X, \tau) \in$ TOP, $\sigma \in \tau$ iff $\sigma \sqsubset_\tau \sigma$ iff $\sigma \in {}^{\tau}\sqsubset_\tau$. Thus $GF(X, \tau) = (X, \tau)$. For any $(X, \sqsubset)$ is in ^{tqp}-HFS, $\mu \sqsubset \rho$ iff there exists $\sigma \in H^X$ such that $\mu \leq \sigma \sqsubset \sigma \leq \rho$ iff $\mu \sqsubset \tau_\sqsubset \rho$. Thus $GF(X, \sqsubset) = (X, \sqsubset)$. This completes the proof.

Theorem 2. 8 The category PROX of H-fuzzy proximity spaces (and proximiy maps) and ^{tsq}-HFS are isomorphic.

[A H-fuzzy proximity (or, simply proximity) on a set X is a map $\delta : H^X x H^X \rightarrow \{0, 1\}$ which satisfies, for any $\mu, \nu, \rho \in H^X$, the following conditions :

P1) $\delta(0,1)=0$.

P2) $\delta(\mu, \rho)=\delta(\rho, \mu)$.

P3) $\delta(\mu, \rho) \vee \delta(\nu, \rho)=\delta(\mu \vee \nu, \rho)$.

P4) If $\delta(\mu, \rho)=0$, there exists $\nu \in H^X$ such that $\delta(\mu, \nu)=0$ and $\delta(\rho, \nu^*)=0$.

P5) $\delta(\mu, \rho)=0$ implies $\mu \leq \rho^*$.

The pair (X, δ) is said to be a H-fuzzy proximity (or, simply proximity) space. If (X, δ) and (Y, η) are proximity spaces, then a map $f : X \rightarrow Y$ is called a proximity map if for any $\mu, \rho \in H^Y$, $\eta(\mu, \rho)=0$ implies $\delta(f^{-1}(\mu), f^{-1}(\rho))=0$ ([1]).]

Proof. For any $(X, \delta) \in$ PROX, define $\sqsubset_\delta$ as follows : $\mu \sqsubset_\delta \rho$ iff $\delta(\mu, \rho^*)=0$. It is clear that $(X, \sqsubset_\delta) \in {}^{tsq}$-HFS. If $f : (X, \delta) \rightarrow (Y, \eta) \in$ PROX and $\mu \sqsubset_\delta \rho$. Then $\eta(\mu, \rho^*)=0$. Since $f \in$ PROX, $\delta(f^{-1}(\mu), f^{-1}(\rho^*))=0$ and hence $f^{-1}(\mu) \sqsubset_\delta f^{-1}(\rho)$. Thus $F :$ PROX$\rightarrow {}^{tsq}$-HFS ($F(X, \delta)=(X, \sqsubset_\delta)$ and $F(f)=f$) is a functor. For any $(X, \sqsubset) \in {}^{tsq}$-HFS, defin a map $\delta_\sqsubset : H^X x H^X \rightarrow \{0. 1\}$ as follows : $\delta_\sqsubset(\mu, \rho)=0$ iff $\mu \sqsubset \rho^*$. It is clear that $(X, \delta_\sqsubset) \in$ PROX. Suppose that $f : (X, \sqsubset) \rightarrow (Y, \ll) \in {}^{tsq}$-HFS and $\delta_\ll(\mu, \rho)=0$. Then $\mu \ll \rho^*$. Since $f \in {}^{tsq}$-HFS, $f^{-1}(\mu) \sqsubset f^{-1}(\rho^*)$ and hence $\delta_\sqsubset(f^{-1}(\mu), f^{-1}(\rho))=0$. Thus $G : {}^{tsq}$-HFS$\rightarrow$PROX ($G(X, \sqsubset)=(X, \delta_\sqsubset)$ and $G(f)=f$) is a functor. For any $(X, \delta) \in$ PROX, $\delta(\mu, \rho)=0$ iff $\mu \sqsubset_\delta \rho^*$ iff $\delta_{\sqsubset_\delta}(\mu, \rho)=0$. Thus $GF(X, \delta)=(X, \delta)$. For any $(X, \sqsubset) \in {}^{tsq}$-HFS, $\mu \sqsubset \rho$ iff $\delta_\sqsubset(\mu, \rho^*)=0$ iff $\mu \sqsubset_{\delta_\sqsubset} \rho$. Thus $FG(X, \sqsubset)=(X, \sqsubset)$. This completes the proof.

Let Ω_X denote the family of all maps $\alpha : H^X \rightarrow H^X$ with the following properties :

A1) $\alpha(0)=0$ and $\mu \leq \alpha(\mu)$ for all $\sqsubset \in H^X$.

A2) $\alpha(\sup \mu_i)=\sup \alpha(\mu_i)$.

Let $f : X \rightarrow Y$ be a map and $\alpha \in \Omega_Y$. Define $f^{-1}(\alpha) : H^X \rightarrow H^X$ as follows : $f^{-1}(\alpha)(\mu)=f^{-1}(\alpha(f(\mu)))$ for any $\mu \in H^X$. If $\alpha \in \Omega_Y$, then $f^{-1}(\alpha) \in \Omega_Y$([6]).

Theorem 2. 9 The category QUNIF of quasi-uniform spaces (and uniform maps) and ^{b}-HFS are isomorphic.

[A H-fuzzy quasi-uniformity (or, simply q-uniformity) on X is a nonempty subset $\tilde{U}$ of Ω_X having the following two properties :

U1) Given $\alpha, \beta \in \tilde{U}$ there exists $\gamma \in \tilde{U}$ with $\gamma \leq \alpha, \beta$.

U2) Given $\alpha \in \tilde{U}$, there exists $\beta \in \tilde{U}$ with $\beta \circ \beta \leq \alpha$.

The pair $(X, \tilde{U})$ is called a H-fuzzy quasi-uniform (or, simply q-uinform) space. Let $\tilde{U}$ and $\tilde{u}$ be q-uniformities on a set X. Then $\tilde{U}$ is said to be coarser than $\tilde{u}$ if for any $\alpha \in \tilde{U}$, there exists $\beta \in \tilde{u}$ such that $\alpha \leq \beta$. In this case we also say that $\tilde{u}$ is coarser than $\tilde{U}$. q-uniformities $\tilde{U}$ and $\tilde{u}$ are said to be equivalent (denoted by $\tilde{U} \cong \tilde{u}$) if $\tilde{U}$ is finer than $\tilde{u}$ and $\tilde{u}$ is finer than $\tilde{U}$. If $(X, \tilde{U})$ and $(Y, \tilde{u})$ are q-uniform spaces, then a map $f : X \rightarrow Y$ is said to be a uniform map if for any $\alpha \in \tilde{u}$, there exists $\beta \in \tilde{U}$ such that $\beta \leq f^{-1}(\alpha)$.]

Proof. By proposition in [5], it is enough to show that Qunif and ^{b}-HFS are equivalent.

a) For any $(X, \tilde{U}) \in$ QUNIF and for each $\alpha \in \tilde{U}$, define a relation $\sqsubset_\alpha$ as follows : $\mu \sqsubset_\alpha \rho$ iff $\alpha(\mu) \leq \rho$. It is clear that $\sqsubset_\alpha$ is biperfect fs. order on X. Let $S=S(\tilde{U})=\{\sqsubset_\alpha : \alpha \in \tilde{U}\}$. Then $(X, S) \in {}^{b}$-HFS.

S1) By U1), it is enough to show that $\alpha \leq \beta$ implies $\sqsubset_\beta \leq \sqsubset_\alpha$. Suppose that $\mu \sqsubset_\beta \rho$. Then $\beta(\mu) \leq \rho$. Since $\alpha \leq \beta$, $\alpha(\mu) \leq \rho$. Thus $\mu \sqsubset_\alpha \rho$.

S2) Take any $\sqsubset \in S$, there exists $\alpha \in \tilde{U}$ such that $\sqsubset_\alpha = \sqsubset$. Since $\tilde{U}$ is q-uniformity, there exists $\beta \in \tilde{U}$ such that $\beta \circ \beta \leq \alpha$. Suppose that $\mu \sqsubset_\alpha \rho$. Then $\alpha(\mu) \leq \rho$. Since $\beta \circ \beta \leq \alpha$, $\beta(\beta(\mu)) \leq \rho$ and hence $\beta(\mu) \sqsubset_\beta$

ρ. Since $\mu \sqsubset_\beta \beta(\mu)$, $\sqsubset_\alpha$ is coarser than $\sqsubset_\beta^2$.

b) If $f:(X, \tilde{U}) \to (Y, \tilde{u}) \in$ QUNIF and $\sqsubset \in S(\tilde{U})$. Then there exists $\alpha \in \tilde{u}$ such that $\sqsubset = \sqsubset_\alpha$. Since $f \in$ QUNIF, there exists $\beta \in \tilde{U}$ such that $f^{-1}(\alpha) \geq \beta$. $\mu f^{-1}(\sqsubset_\alpha)\rho$ iff $f(\mu) \sqsubset_\alpha (f(\rho^*))^*$ iff $\alpha(f(\mu)) \leq (f(\rho^*))^*$. Then $f^{-1}(\alpha(f(\mu)) \leq f^{-1}(f(\rho^*))^*$ and hence $f^{-1}(\alpha)(\mu) \leq \rho$. Since $\beta \leq f^{-1}(\alpha)$, $\beta(\mu) \leq \rho$. Thus $\mu \sqsubset_\beta \rho$. F: QUNIF$\to$ ^{b}-HFS$(F(X, \tilde{U}) = (X, S(\tilde{U}))$ and $F(f) = f)$ is a functor.

c) For any $(X, S) \in$ ^{b}-HFS and for each $\sqsubset \in S$, let $\alpha_\sqsubset(\mu) = \inf\{\rho : \mu \sqsubset \rho\}$. It is clear that $\alpha_\sqsubset \in \Omega_X$. Then $\tilde{U} = \tilde{U}(S) = \{\alpha_\sqsubset : \sqsubset \in S\}$ is a q-uniformity on X.

U1) It follows immediately by the definition.

U2) Take any $\alpha \in \tilde{U}$ there exists $\sqsubset \in S$ such that $\alpha = \alpha_\sqsubset$. Since $\sqsubset \in S$, there exists $\ll \in S$ such that $\sqsubset$ is coarser than $\ll^2$. It is clear that for each $\mu \in H^X$, $\alpha(\mu) \sqsubset \mu$. Thus $\alpha(\mu) \ll \sigma \ll \mu$ for some $\sigma \in H^X$ and hence $\alpha_\ll \circ \alpha_\ll \leq \alpha$

d) If $f:(X, S) \to (Y, T) \in$ ^{b}-HFS and let $\alpha \in \tilde{U}(T)$. Then there exists $\sqsubset \in T$ such that $\alpha = \alpha_\sqsubset$. Since $f \in$ ^{b}-HFS, there exists $\ll \in S$ such that $f^{-1}(\sqsubset)$ is coarser than $\ll$. Take any $\mu \in H^X$, $f(\mu) \sqsubset \alpha_\sqsubset(f(\mu))$ and hence $\mu \ll f^{-1}(\alpha_\sqsubset)(\mu)$. Thus $\alpha_\ll)(\mu) \leq f^{-1}(\alpha_\sqsubset)(\mu)$ and hence $\alpha_\ll \leq f^{-1}(\alpha_\sqsubset)$. G : ^{b}-HFS$\to$QUNIF $(G(X,S) = (X, \tilde{U}(S))$ and $G(f) = f)$ is a functor.

e) For any $(X, \tilde{U}) \in$ QUNIF, $\tilde{U} \cong \tilde{U}(S(\tilde{U}))$. If $\alpha \in \tilde{U}(S(\tilde{U}))$, then there exists $\sqsubset \in S(\tilde{U})$ such that $\alpha = \alpha_\sqsubset$. Since $\sqsubset \in S(\tilde{U})$, there exists $\beta \in \tilde{U}$ such that $\sqsubset = \sqsubset_\beta$. Then $\beta \leq \alpha$. Indeed, for each $\mu \in H^X$, $\mu \sqsubset_\beta \alpha_{\sqsubset_\beta}(\mu)$ iff $\beta(\mu) \leq \alpha_{\sqsubset_\beta}(\mu)$. Thus $\beta \leq \alpha$. If $\alpha \in \tilde{U}$, then there exists $\beta \in \tilde{U}$ such that $\beta \circ \beta \leq \alpha$. Then for each $\mu \in H^X$, $\mu \leq \beta(\mu) \sqsubset_\beta \alpha(\mu)$ and hence $\gamma_{\sqsubset_\beta}(\mu) \leq \alpha(\mu)$. Thus $\gamma_{\sqsubset_\beta} \leq \alpha$.

f) For any $(X, S) \in$ ^{b}-HFS, $S \cong S(\tilde{U}(S))$. If $\sqsubset \in S(\tilde{U}(S))$, then there exists $\alpha \in \tilde{U}(S)$ such that $\sqsubset = \sqsubset_\alpha$. Since $\alpha \in \tilde{U}(S)$, there exists $\ll \in S$ such that $\alpha = \alpha_\ll$. $\mu \sqsubset \rho$ iff $\mu \sqsubset_{\alpha_\ll} \rho$ iff $\alpha_\ll(\mu) \leq \rho$ and hence $\mu \ll \alpha_\ll \leq \rho$. Thus $\mu \ll \rho$. If $\sqsubset \in S$, then there exists $\ll \in S$ such that $\sqsubset$ is coarser than $\ll^2$. Suppose $\mu \sqsubset \rho$. Then $\mu \ll \sigma \ll \rho$ for some $\sigma \in H^X$. Then $\alpha_\sqsubset(\mu) \leq \sigma \sqsubset \rho$ and hence $\mu \sqsubset_{\alpha_\sqsubset} \rho$. This completes the proof.

Let X be a set and $\alpha \in \Omega_X$. For each f. set μ in X, let $\alpha^{-1}(\mu) = \{\rho : \alpha(\rho^*) \leq \mu^*\}$. Then $\alpha^{-1} \in \Omega_X$ ([6]).

Remark 1) Let $\sqsubset$ be a biperfect fs. order on a set X. Then $\alpha_{\sqsubset^c} = \alpha_\sqsubset^{-1}$ and hence if $\sqsubset$ is symmetrical then $\alpha_\sqsubset = \alpha_\sqsubset^{-1}$.

2) let $\alpha \in \Omega_X$. Then $\sqsubset_{\alpha^c} = \sqsubset_{\alpha^{-1}}$ and hence if $\alpha = \alpha^{-1}$, then $\sqsubset_\alpha$ is symmetrical.

Proof. 1) Let μ be a f. set in X. $\alpha_{\sqsubset^c}(\mu) = \inf\{\rho : \mu \sqsubset^c \rho\} = \inf\{\rho : \rho^* \sqsubset \mu^*\} = \alpha_\sqsubset^{-1}(\mu)$. The remaing part is clear.

2) Let μ and ρ be f. sets in X. $\mu \sqsubset_{\alpha^{-1}} \rho$ iff $\alpha^{-1}(\mu) \leq \rho$ iff $\alpha(\rho^*) \leq \mu^*$ iff $\mu \sqsubset_{\alpha^c} \rho$. The remaing part is clear.

The following is immediate from the above remark and (2.9).

Theorem 2. 10 The category UNIF of uniform spaces (and uniform maps) and ^{sb}-HFS are isomorphic. [A q-uniform space $(X, \tilde{U})$ is said to be uniform space if for each $\alpha \in \tilde{U}$, $\alpha = \alpha^{-1}$.]

Remark It is immediate from the above theorem, (2. 6. 4) and (2. 9) that UNIF is coreflectiv in QUNIF.

Theorem 2. 12 Let ^{a} be a symmetrical elementary operation. Then ^{sa}-HFS is closed under the formation of initial sources in ^{a}-HFS.

Proof. Suppose that $(f_i : (X, S) \to (X_i, S_i))_{i \in I}$ is an initial source in ^{a}-HFS. Since ^{sa}-HFS is coreflective in ^{a}-HFS, $S = (\cup f_i^{-1}(S_i))^{ga}$. Let $\sqsubset = (\cup_{i \in F} f_i^{-1}(\sqsubset_i))^a$, where $\sqsubset_i \in S_i (i \in F)$ and F is an nonempty subset of I.

$$\sqsubset^{sa} = ((\bigcup_{i \in F} f_i^{-1}(\sqsubset_i)) \cup (\bigcup_{i \in F} f_i^{-1}(\sqsubset_i))^c)^a$$
$$= ((\bigcup_{i \in F} f_i^{-1}(\sqsubset_i)) \cup (\bigcup_{i \in F} f_i^{-1}(\sqsubset_i^c)))^a.$$

Since ^{a} is symmetrical and $(X_i, S_i) \in {}^{sa}$-HFS, $\sqsubset^{sa} = \sqsubset$. This completes the proof.

Remark It is immediate from the above theorem, (2. 9) and theorem in [11] that UNIF is bireflective in QUINF.

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