Journal of the Korean
Statistical Society
Vol. 22, No. 2, 1993

On The Dichotomy of Stationary and Ergodic Probability Measures

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ABSTRACT

The dichotomy of absolute continuity and singularity for a pair of stationary and ergodic measures (one of which need not be ergodic) is obtained using the ergodic decomposition theorem. The known fact that two different stationary and ergodic measures are mutually singular is obtained as a corollary of our result. An example of a pair of stationary-ergodic measures enjoying the dichotomy is presented.

KEYWORDS: Absolute Continuity, Dichotomy, Ergodic Decomposition Theorem, Sigularity, Stationary and Ergodic Probability Measures.

1. INTRODUCTION

Suppose that two probability measures P and Q are given on a measurable space $(\Omega, \mathcal{F})$. Q is called *absolutely continuous* with respect to P (denoted by $Q << P$) if $Q(A) = 0$ whenever $P(A) = 0$ for $A \in \mathcal{F}$. If $Q << P$ and $P << Q$, these measures are called *equivalent* ($Q \sim P$). We say that Q and P are (mutually) *singular* or *orthogonal* ($Q \perp P$) if there exists a set $B \in \mathcal{F}$ such that $Q(B) = 0$ and $P(B^c) = 0$. There are some cases that Q and P are either absolutely continuous or singular: the *dichotomy*. It is of course possible that Q and P are neither absolutely continuous nor singular. For example, let Q and P be singular on $(\Omega, \mathcal{F})$, and let $R = (P+Q)/2$. Then neither $R << P$ nor $R \perp P$.

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Kakutani (1948) obtained the dichotomy theorem for infinite product probability measures using the Hellinger distance. Hajek (1958) and Feldman (1958) independently proved the dichotomy property for Gaussian measures, and Gihman and Skorohod (1966), and Brown (1971) obtained the dichotomy for Poisson processes. Also it is known that two probability measures induced by stationary and ergodic processes are either same or singular (see Breiman (1968), Corollary 6.24, for example). For arbitrary probability measures, Kabanov, Liptser and Shiryayev (1977) established some necessary and sufficient conditions for absolute continuity and singularity using the martingale theory.

In this note, the dichotomy theorem for a pair of stationary and ergodic measures (one of which need not be ergodic) is obtained using the ergodic decomposition theorem and the ergodic convergence theorem. The Corollary 6.24 in Breiman (1968) which states that two different stationary and ergodic measures are mutually singular is obtained as a corollary of our result. An example of a pair of stationary and ergodic measures, one of which is not ergodic, enjoying the dichotomy is constructed in the last section.

2. MAIN RESULT

Theorem 1. Let Ω be a separable metric space, and $\mathcal{F}$ be a Borel field generated by Ω. Let P be a stationary measure on $(\Omega, \mathcal{F})$ with transformation operator T, and Q be a stationary-ergodic measure on $(\Omega, \mathcal{F})$. Then the dichotomy arises: either $Q \perp P$ or $Q << P$.

Proof. Let Λ be a metric space (with the Prohorov distance) which is the class of all stationary-ergodic probability measures P_{erg} on $(\Omega, \mathcal{F})$, and $\mathcal{F}_\Lambda$ be a Borel field generated by Λ. Then the ergodic decomposition theorem (see Gray (1988), Section 7.4, for reference) asserts the existence of a measure λ on $(\Lambda, \mathcal{F}_\Lambda)$ such that

$$P(A) \;=\; \int_\Lambda P_{erg}(A)\; \lambda(dP_{erg}), \tag{2.1}$$

for $A \in \mathcal{F}$ and $P_{erg} \in \Lambda$.

Let θ index all stationary-ergodic probability measures on $(\Omega, \mathcal{F})$ in one-to-one manner (i.e., $P_\theta(A) = P_{\theta'}(A)$ for all $A \in \mathcal{F}$ iff $\theta = \theta'$), and Θ be the class of all such θ's. This indexation is of course identifiable in the sense that $\theta \neq \theta'$ implies that there exists a set $B \in \mathcal{F}$ such that $P_\theta(B) \neq P_{\theta'}(B)$. Thus the parametrization of *all* P_{erg} on $(\Omega, \mathcal{F})$ by an index set Θ is always possible by giving indexes to all equivalent classes which consist of a partition of Λ. Then (2.1) is rewritten by

$$P(A) \;=\; \int_\Theta P_\theta(A)\; \lambda(d\theta), \tag{2.2}$$

for $A \in \mathcal{F}$ and $\theta \in \Theta$. Then, by the ergodic decomposition theorem again, their exists a measurable function $\theta(\omega): \Omega \longrightarrow \Theta$ such that, for $A \in \mathcal{F}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} 1_A(T^i\omega) = P_{\theta(\omega)}(A) \quad a.e.\ [P]. \tag{2.3}$$

Let

$$B = \{\omega : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} 1_{A_k}(T^i\omega) = Q(A_k),\ k \geq 1\}, \tag{2.4}$$

where a collection of sets $\{A_k\}_{k\geq 1}$ countably generates $\mathcal{F}$, by the separability of Ω. Then $Q(B) = 1$ by the ergodic convergence theorem. We see from (2.3) and (2.4) that

$$\begin{aligned} P(B) &= P\{\omega : P_{\theta(\omega)}(A_k) = Q(A_k),\ \ k \geq 1\} \\ &= P\{\omega : P_{\theta(\omega)} = Q\}, \end{aligned} \tag{2.5}$$

since $\{A_k\}_{k\geq 1}$ countably generates $\mathcal{F}$. Take a $\theta^* \in \Theta$ such that $P_{\theta^*} = Q$, since Q is a stationary-ergodic measure on $(\Omega, \mathcal{F})$. Then by the one-to-one property of P_θ, we have from (2.5) that

$$\begin{aligned} P(B) &= P\{\omega : P_{\theta(\omega)} = P_{\theta^*}\} \\ &= P\{\omega : \theta(\omega) = \theta^*\} \\ &= \lambda\{\theta : \theta = \theta^*\}. \end{aligned} \tag{2.6}$$

If (2.6) is equal to 0, then $P(B) = 0$ and $Q \perp P$. Thus suppose (2.6) be positive. Then, for any $A \in \mathcal{F}$, it follows from (2.2) that

$$\begin{aligned} P(A) &= \int_{\{\theta=\theta^*\}} P_\theta(A)\ \lambda(d\theta) + \int_{\{\theta\neq\theta^*\}} P_\theta(A)\ \lambda(d\theta) \\ &= \lambda(\theta = \theta^*)Q(A) + \int_{\{\theta\neq\theta^*\}} P_\theta(A)\ \lambda(d\theta). \end{aligned} \tag{2.7}$$

Since the second term of (2.7) is nonnegative and $\lambda(\theta = \theta^*) > 0$, $P(A) = 0$ implies $Q(A) = 0$, and consequently $Q << P$. Therefore, the dichotomy is obtained, according to $P(B) = 0$ or positive.

Theorem 2. Under the same condition of Theorem 1, if P is stationary and ergodic, then the followings are equivalent:

(1) $P \sim Q$,
(2) P and Q are not singular,
(3) $P = Q$,

(4) $P(B) > 0$ (actually $P(B) = 1$).

Proof. First, it is enough to show $P << Q$ when $P(B) > 0$, where B is same to (2.4). Since B is an invariant event and P is a stationary-ergodic measure, $P(B) > 0$ implies $P(B) = 1$. That is, $\lambda(\theta = \theta^*) = 0$ and consequently the second term of (2.7) vanishes. Thus $Q(A) = 0$ implies $P(A) = 0$ in (2.7), and so that $P << Q$. Therefore, incoporating with Theorem 1, we have $P \perp Q$ iff $P(B) = 0$, and $P \sim Q$ iff $P(B) > 0$ (actually $P(B) = 1$). Moreover, since $P(B) = 1$ iff $\lambda(\theta = \theta^*) = 1$, $P(A) = Q(A)$ for any $A \in \mathcal{F}$ from (2.7). This completes the proof.

The Theorem 2 implies that two different stationary-ergodic measures are mutually singular (Corollary 6.24 in Breiman (1968)), and two equivalent stationary-ergodic measures are same.

3. AN EXAMPLE

A pair of stationary-ergodic measures, one of which is not ergodic, enjoying the dichotomy is constructed in this section.

Let m_1 and m_2 are two different stationary-ergodic measures, and

$$P = \alpha m_1 + (1 - \alpha) m_2, \quad 0 < \alpha < 1.$$

Then P is stationary but not ergodic (see Gray (1988), p. 212), and $m_1 << P$ and $m_2 << P$.

Now consider a stationary-ergodic measure Q which is not equal to m_1 and m_2. Then $Q \perp m_1$ and $Q \perp m_2$ by Theorem 2. So there exist sets S, $U \in \mathcal{F}$ such that $Q(S) = 0$, $m_1(S) = 1$ and $Q(U) = 0$, $m_2(U) = 1$. Thus $Q(S \cup U) = 0$ and $P(S \cup U) = 1$, and consequently $Q \perp P$.

Therefore, if $Q = m_1$ or $Q = m_2$ then $Q << P$, and if $Q \neq m_1$ and $Q \neq m_2$ then $Q \perp P$.

ACKNOWLEDGEMENT

The author thanks Professor Andrew Barron for helpful discussions.

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