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Unit Root Test for Temporally Aggregated Autoregressive Process

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ABSTRACT

Unit root test for temporally aggregated first order autoregressive process is considered. The temporal aggregate of first order autoregression is an autoregressive moving average of order (1,1) with moving average parameter being function of the autoregressive parameter. One-step Gauss-Newton estimators are proposed and are shown to have the same limiting distribution as the ordinary least squares estimator for unit root when complete observations are available. A Monte-Carlo simulation shows that the temporal aggregation have no effect on the size. The power of the suggested test are nearly the same as the powers of the test based on complete observations.

KEYWORDS: Unit root test, autoregressive process, temporal aggregation, autoregressive moving average, Gauss-Newton procedure.

1. INTRODUCTION

Let x_t be a first order autoregressive process (AR(1))

$$x_t = \phi\, x_{t-1} + \epsilon_t, \tag{1.1}$$

where $\{\epsilon_t\}$ is an iid $(0, \sigma_{\epsilon\epsilon})$ sequence and x_0 is a fixed initial value. When ϕ is one, $\{x_t\}$ is called a random walk and the process is nonstationary. When $|\phi| < 1$, $\{x_t\}$ is asymptotically stationary. We are interested in testing for unit root $\phi = 1$.

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In real world, there are situations in which only temporal aggregate of $\{x_t\}$ are available. For example, time unit for $\{x_t\}$ in (1.1) is day and we only observe weekly totals. Let n and m be positive integers. The temporal aggregate $\{y_i\}$ of $\{x_t\}$ is

$$y_1 = x_1 + \cdots + x_m, \cdots, y_n = x_{nm-m+1} + \cdots + x_{nm}. \tag{1.2}$$

There has been much interest in testing unit root hypothesis $H_0 : \phi = 1$ since Fuller(1976, section 8.5), and Dickey and Fuller(1979). They derived the limiting distribution of the ordinary least squares estimator of ϕ in model (1.1). They also prepared percentiles of the limiting distribution. Higher order autoregressive process with unit roots are studied by Hasza and Fuller(1979), Dickey et. al.(1984), Ahtola and Tiao(1987), and Chan and Wei(1988). Autoregressive moving average(ARMA) with one unit root are studied by Said and Dickey(1984, 1985), Hall(1989), Shin(1990), and Shin and Fuller(1992). Shin and Pantula(1993) studied tests for unit root in autoregressive processes with systematic but incomplete sampling. Shin and Sarkar(1993) investigated the maximum likelihood estimation for unit root in autoregressive process disturbed by moving average.

Some results for temporally aggregated time series are found in the literature. See Wei and Stram(1990) and references therein. Some authors are interested in the estimation of original process $\{x_t\}$ from the aggregated observation $\{y_i\}$. Nijman and Palm(1990) studied parameter identification and estimation of stationary ARMA(2,1) and AR(2) model when systematically incomplete observations $\{x_{im}\}_{i=1}^{n}$ are available.

It is well known that the temporal aggregate is an autoregressive moving average of order (1,1) with autoregressive parameter $\rho = \phi^m$. We derive an explicit relationship between the autoregressive parameter and the moving average coefficient.

In section 3, we propose test statistics for testing unit root when the observations are the temporal aggregate $\{y_i\}_1^n$ of the underlying process $\{x_t\}_1^{nm}$, which take full advantage of the relationship between the autoregressive parameter and the moving average parameter. The statistics have the standard limiting distributions of Dickey and Fuller(1979). The test statistics are one-step Gauss-Newton estimator of ρ and t-statistic of it.

In section 4, the effect of the aggregation is investigated via Monte Carlo simulation. Empirical sizes of suggested test statistics are compared with the Dickey-Fuller test statistics which are appropriate when complete observations $\{x_t\}_1^{nm}$ are available. The empirical powers of the suggested test statistics are slightly less than those of Dickey-Fuller test statistics based on complete observation $\{x_t\}_1^{nm}$. As for the unit root test, effects of aggregation on size and power are shown to be minor.

2. MODELS FOR TEMPORAL AGGREGATE

It is well known that the temporally aggregated AR(1) series $\{y_i\}$ has ARMA(1,1) representation. We derive an explicit relationship between the autoregressive parameter and the moving average parameter. We take advantage of the relationship in estimating autoregressive parameter. From (1.1) we have

$$x_t = \phi^m x_{t-m} + \phi^{m-1}\epsilon_{t-m+1} + \cdots + \phi\, \epsilon_{t-1} + \epsilon_t. \tag{2.1}$$

From (2.1), adding $x_{im-m+1}, \cdots, x_{im}$, we get

$$y_i = \rho\, y_{i-1} + u_i,\ i = 1,\ \ldots,\ n,$$

where

$$\begin{aligned} u_i &= s_1(\phi)\epsilon_{im} + s_2(\phi)\epsilon_{im-1} + \ldots\ + s_m(\phi)\epsilon_{im-m+1} \\ &\quad + c_1(\phi)\epsilon_{im-m} + \ldots\ + c_{m-1}(\phi)\epsilon_{im-2m+2}, \\ s_k(\phi) &= 1 + \phi + \ldots\ + \phi^{k-1},\ k = 1,\ 2,\ \ldots,\ m, \end{aligned}$$

and

$$c_k(\phi) = \phi^k + \ldots\ + \phi^{m-1},\ k = 1,\ 2,\ \ldots,\ m-1.$$

Since y_0 does not affect the limiting distribution of the estimator, we assume that y_0 is zero. Note that the autocovariance function $\gamma_h(\phi) = cov(u_1,\ u_{1+h})$ is

$$\gamma_0(\phi) = \sigma_{\epsilon\epsilon}\{\sum_1^m s_k^2(\phi) + \sum_1^{m-1} c_k^2(\phi)\}, \tag{2.2}$$

$$\gamma_1(\phi) = \sigma_{\epsilon\epsilon} \sum_1^{m-1} s_k(\phi)c_k(\phi), \tag{2.3}$$

and $\gamma_h(\phi) = 0$ for $h \geq 2$. Therefore $\{u_i\}$ is a first order moving average and we can write

$$y_i = \rho\, y_{i-1} + e_i + \beta\, e_{i-1},\ i = 1,\ \ldots,\ n \tag{2.4}$$

for some iid $(0,\ \sigma^2)$ sequence $\{e_i\}$ and a real number β, where $\rho = \phi^m$. The moving average coefficient β is determined by comparing (2.2)-(2.3) and autocovariances of $(e_i + \beta\, e_{i-1})$ in (2.4),

$$\sigma^2(1 + \beta^2) = \gamma_0(\phi),\ \sigma^2\beta = \gamma_1(\phi). \tag{2.5}$$

There are two values of β which satisfies (2.5). We choose β in the invertibility region. Hence

$$\beta = [1 - \{1 - 4\rho_1^2(\phi)\}^{1/2}]/\{2\rho_1(\phi)\}, \tag{2.6}$$

where $\rho_1(\phi) = \gamma_1(\phi)/\gamma_0(\phi)$. Note that β in (2.6) is a function of $\rho = \phi^m$ and lie in (-1, 1) because $|\rho_1(\phi)| < 1/2$ for all $\phi \in (-1, 1)$.

Estimation for model (2.4) subject to (2.5) is a constarined nonlinear estimation of ARMA model. Pagano(1974), Sakai and Arase(1979), and Shin(1993) investigated estimation of ARMA model subject to nonlinear restriction similar to (2.5).

We next consider the model with mean μ. If $\{x_t\}$ is generated by

$$x_t = \mu_1 + \phi\ x_{t-1} + \epsilon_t \tag{2.7}$$

then $\{y_i\}$ have the following representation

$$y_i = \mu + \rho\ y_{i-1} + e_i + \beta\ e_{i-1}, \tag{2.8}$$

where

$$\mu = m\mu_1(1 + \phi + \dots + \phi^{m-1}).$$

In (2.8), the parameters ρ and β are the same as those in (2.4).

3. ESTIMATION AND TESTING OF UNIT ROOT

We propose test statistics for testing $H_0 : \phi = 1$ in model (1.1) and in model (2.7) based on the temporal aggregate $\{y_i\}_1^n$. The test statistics are one-step Gauss-Newton estimators which take advantage of the relation (2.5) of the moving average term in (2.4) and (2.8).

The temporal aggregate $\{y_i\}$ for model (1.1) and (2.7) follow autoregressive moving average model of order (1,1)

$$y_i = \rho\ y_{i-1} + e_i + \beta\ e_{i-1},\ i = 1, \dots, n, \tag{3.1}$$

and

$$y_i = \mu + \rho\ y_{i-1} + e_i + \beta\ e_{i-1},\ i = 1, \dots, n, \tag{3.2}$$

respectively. Testing unit root $\phi = 1$ in model (1.1) and (2.7) can be performed by testing $\rho = 1$ in (3.1) and (3.2), respectively. Since there are moving average terms in (3.1) and (3.2), estimation is a nonlinear problem.

3.1. Model without intercept

We first consider model (3.1). Our procedure is an one-step Gauss-Newton procedure which is similar to Fuller(1976, section 8.3). Let

$$e_i(\rho) = y_i - \rho\, y_{i-1} - \beta\, e_{i-1}(\rho) \tag{3.3}$$

and

$$e_{\rho i}(\rho) = \partial e_i(\rho)/\partial \rho = -y_{i-1} - \beta_\rho e_{i-1}(\rho) - \beta\, e_{\rho,i-1}(\rho), \tag{3.4}$$

$i = 1, \ldots, n$, where $\beta_\rho = d\beta/d\rho$, $e_0(\rho) = y_0 = e_{\rho,0}(\rho) = 0$. One-step Gauss-Newton estimator $\hat{\rho}$ is given by

$$\hat{\rho} = \bar{\rho} - \sum_1^n e_{\rho i}(\bar{\rho}) e_i(\bar{\rho}) / \sum_1^n e_{\rho i}^2(\bar{\rho}), \tag{3.5}$$

where $\bar{\rho}$ is an initial estimator. Once $\bar{\rho}$ is given, $\bar{\phi}$ is obtained from $\rho = \phi^m$. If m is odd, $\bar{\phi} = \bar{\rho}^{1/m}$. If m is even, $\bar{\phi} = \bar{\rho}^{1/m}$ for $\bar{\rho} > 0$ and $\bar{\phi} = 0$ for $\bar{\rho} \le 0$. When $\bar{\phi}$ is available, we can find $\bar{\beta}$ and $\bar{\beta}_\rho$ from (2.6).

As an initial estimator of ρ, we may use $\bar{\rho} = 1$. Alternatively, the instrument variable estimator $\bar{\rho} = \bar{\rho}_{iv} = \sum_3^n y_i y_{i-2} / \sum_3^n y_{i-1} y_{i-2}$ may be used. Hall(1989) showed that $n(\bar{\rho}_{iv} - 1)$ has the limiting distribution (3.10) in Theorem 1 when $\rho = 1$.

The t-statistic for $H_0 : \rho = 1$ in model (3.1) is

$$\hat{\tau} = (\hat{\rho} - 1)/\{s^2 / \sum_1^n e_{\rho i}^2(\hat{\rho})\}^{1/2}, \tag{3.6}$$

where $s^2 = \sum_1^n e_i^2(\hat{\rho})/(n-2)$.

It is interesting to note that, when $m = 1$, β becomes zero and $\sum e_{\rho i}^2(\hat{\rho})$ becomes $\sum y_{i-1}^2$. Therefore $\hat{\tau}$ becomes the Dickey and Fuller test statistics tau for AR(1) model. Hence, it can be said that $\hat{\tau}$ in (3.6) is a generalization of the Dickey and Fuller test statistic tau in terms of the temporal aggregation.

3.2. Model with intercept

We next consider model (3.2) which contains mean μ. Let $(\bar{\rho}_\mu, \bar{\mu})$ be an initial estimator. One-step Gauss-Newton estimator $(\hat{\rho}_\mu, \hat{\mu})$ is given by

$$\begin{bmatrix} \hat{\rho}_\mu \\ \hat{\mu} \end{bmatrix} = \begin{bmatrix} \bar{\rho}_\mu \\ \bar{\mu} \end{bmatrix} - \begin{bmatrix} \sum e_{\rho i}^2(\bar{\rho}_\mu, \bar{\mu}) & \sum e_{\rho i}(\bar{\rho}_\mu, \bar{\mu}) e_{\mu i}(\bar{\rho}_\mu, \bar{\mu}) \\ symmetric & \sum e_{\mu i}^2(\bar{\rho}_\mu, \bar{\mu}) \end{bmatrix}^{-1} \times \begin{bmatrix} \sum e_{\rho i}(\bar{\rho}_\mu, \bar{\mu}) e_i(\bar{\rho}_\mu, \bar{\mu}) \\ \sum e_{\mu i}(\bar{\rho}_\mu, \bar{\mu}) e_i(\bar{\rho}_\mu, \bar{\mu}) \end{bmatrix} \tag{3.7}$$

where

$$e_i(\rho, \mu) = y_i - \mu - \rho\, y_{i-1} - \beta\, e_{i-1}(\rho, \mu),$$
$$e_{\rho i}(\rho, \mu) = \partial e_i(\rho, \mu)/\partial \rho = -y_{i-1} - \beta_\rho e_{i-1}(\rho, \mu) - \beta\, e_{\rho,i-1}(\rho, \mu),$$

$$e_{\mu i}(\rho,\ \mu) = \partial e_i(\rho,\ \mu)/\partial\mu = -1 - \beta\ e_{\mu,i-1}(\rho,\ \mu),\ i = 1,\ \ldots,\ n,$$

and $y_0 = e_0(\rho,\ \mu) = e_{\rho 0}(\rho,\ \mu) = e_{\mu 0}(\rho,\ \mu) = 0$. From (3.7), $\hat{\rho}_\mu$ is given by

$$\hat{\rho}_\mu = \bar{\rho}_\mu - \bar{a}/\bar{b}, \tag{3.8}$$

where

$$\bar{a} = [\sum\ e^2_{\mu i}(\bar{\rho}_\mu\ ,\ \bar{\mu})][\sum e_{\rho i}(\bar{\rho}_\mu\ ,\ \bar{\mu})e_i(\bar{\rho}_\mu\ ,\ \bar{\mu})] - [\sum\ e_{\rho i}(\bar{\rho}_\mu\ ,\ \bar{\mu})e_{\mu i}(\bar{\rho},\bar{\mu})][\sum\ e_{\mu i}(\bar{\rho}_\mu\ ,\ \bar{\mu})e_i(\bar{\rho}_\mu\ ,\ \bar{\mu})]$$

and

$$\bar{b} = [\sum\ e^2_{\rho i}(\bar{\rho}_\mu\ ,\ \bar{\mu})][\sum\ e^2_{\mu i}(\bar{\rho}_\mu\ ,\ \bar{\mu})] - [\sum\ e_{\rho i}(\bar{\rho}_\mu\ ,\ \bar{\mu})\sum\ e_{\mu i}(\bar{\rho}_\mu\ ,\ \bar{\mu})]^2.$$

As an initial estimator $(\bar{\rho}_\mu\ ,\ \bar{\mu})$, we may use $(\bar{\rho}_\mu\ ,\ \bar{\mu}) = (1,\ 0)$. Alternatively we may use the instrument variable estimator $(\bar{\rho}_\mu\ ,\ \bar{\mu}) = (\hat{\rho}_{\mu,iv},\ \hat{\mu}_{iv})$ given by

$$\begin{bmatrix}\hat{\rho}_{\mu,iv}\\ \hat{\mu}_{iv}\end{bmatrix} = \begin{bmatrix}\sum_3^n y_{i-2}y_{i-1} & \sum_3^n y_{i-2}\\ \sum_3^n y_{i-1} & (n-3)\end{bmatrix}^{-1}\begin{bmatrix}\sum_3^n y_{i-2}y_i\\ \sum_3^n y_i\end{bmatrix}.$$

Hall(1989) showed that $n(\hat{\rho}_{\mu,iv} - 1)$ has the limiting distribution (3.12) in Theorem 1 when $(\rho,\ \mu) = (1,\ 0)$.

The t-statistic for $H_0 : \rho = 1$ is

$$\hat{\tau}_\mu = (\hat{\rho}_\mu - 1)/(s^2_\mu/\ c_{11})^{1/2}, \tag{3.9}$$

where $s^2_\mu = \sum_1^n e^2_i(\hat{\rho},\ \hat{\mu})/(n-3)$ and c_{11} is the (1,1) element of the (2×2) inverted matrix in (3.7) evaluated at $(\hat{\rho}_\mu,\ \hat{\mu})$ instead of $(\bar{\rho}_\mu,\ \bar{\mu})$.

The limiting distribution of the statistics $\hat{\rho}$, $\hat{\tau}$, $\hat{\rho}_\mu$ and $\hat{\tau}_\mu$ are given in Theorem 1.

Theorem 1. i) Consider model(1.1) with $\phi = 1$ or equivalently model (2.4) with $\rho = 1$. Let $\bar{\rho}$ be an estimator which satisfies $(\bar{\rho} - 1) = o_p(n^{-1/2})$ and let $(\hat{\rho},\ \hat{\tau})$ be defined by (3.5) and (3.6). The limiting distribution of $n(\hat{\rho} - 1)$ and $\hat{\tau}$ are given by

$$n(\hat{\rho} - 1) \to_d 2^{-1}G^{-1}(T^2 - 1) \tag{3.10}$$

and

$$\hat{\tau} \to_d 2^{-1}G^{-1/2}(T^2 - 1), \tag{3.11}$$

where $(G,\ T) = (\sum_1^\infty r_i^2 a_i^2,\ \sum_1^\infty 2^{1/2} r_i a_i)$, $r_i = 2(-1)^{i+1}/\{(2i-1)\pi\}$, $\{a_i\}$ is an NID(0,1) sequence, $\to_d$ denotes convergence in distribution.

ii) Consider model (2.7) with $(\phi,\ \mu_1) = (1,\ 0)$ or equivalently model (2.8) with $(\rho,\ \mu) = (1,\ 0)$. Let $(\bar{\rho}_\mu,\ \bar{\mu})$ be an estimator which satisfy $(\bar{\rho} - 1) = o_p(n^{-1/2})$, $\bar{\mu} = o_p(1)$ and let $(\hat{\rho}_\mu,\ \hat{\tau}_\mu)$ be defined by (3.8) and (3.9). The limiting distribution of $n(\bar{\rho}_\mu - 1)$ and $\hat{\tau}_\mu$ are given by

$$n(\hat{\rho}_\mu - 1) \rightarrow_d 2^{-1}(G - W^2)^{-1}(T^2 - 1 - 2TW) \tag{3.12}$$

and

$$\hat{\tau}_\mu \rightarrow_d 2^{-1}(G - W^2)^{-1/2}(T^2 - 1 - 2TW), \tag{3.13}$$

where $W = \sum_1^\infty 2^{1/2} r_i^2 a_i$.

Proof. Proof is relegated to Appendix.

4. NUMERICAL RESULTS

We study the performance of our estimator using a Monte Carlo simulation. By the subroutine RNNOA of IMSL, a set of $n \times m$ number of NID(0, 1) random number $\{\epsilon_t\}_1^{nm}$ is generated. Then $\{y_i\}_1^n$ of (1.2) is computed with $\phi = 1,\ 0.99^{1/m},\ 0.95^{1/m}$ and $0.90^{1/m}$, for $m = 1$, 2, 4 and 12. From the observation $\{y_i\}_1^n$, the statistics $\hat{\rho}$, $\hat{\tau}$, $\hat{\rho}_\mu$, $\hat{\tau}_\mu$ are computed, based on initial estimator $\bar{\rho} = 1$, $(\bar{\rho},\ \bar{\mu}) = (1,\ 0)$. Results based on instrument variable estimators as initial estimators are almost same and are not reported here.

When m is unity, we have complete observations because $y_i = x_i$. For $m = 2$, series length n is taken to be 25 and 125. For $m = 4$, series length n is taken to be 25 and 100. For $m = 7$, series length n is taken to be 25 and 50. For $m = 12$, n has the values of 25 and 50. In Table 1, we report the empirical sizes and powers of the unit root test $H_0 : \rho = 1$ *vs* $H_1 : \rho < 1$ of size 0.05 based on 5000 samples for each $(n,\ m,\ \rho)$. The cutoff values are available in Fuller(1976, pp. 371, 373). For $n = 125$, however, the cutoff value is interpolated from those for $n = 100$ and $n = 250$.

The most interesting result is that the sizes based on temporal aggregate are almost same as those based on complete observation and the powers based on temporal aggregate are not much different from those based on complete observation. Especially, when m is two or $(n \times m)$ is large, the two powers are almost same. However, when m is seven and $(n \times m)$ is not large, the powers based on temporal aggregate get a little bit smaller than those based on full observation as ρ moves away from one. From this numerical experiment, we can conclude that the effects of temporal aggregation on power are minor, especially if we fit the aggregated series taking into consideration of the relation between the parameters.

Table 1 Empirical sizes and powers of test

$H_0 : \rho = 1 \; vs \; H_1 : \rho < 1$ of size 0.05

	$n \times m = 50$								$n \times m = 250$							
	$\hat{\rho}$		$\hat{\tau}$		$\hat{\rho}_\mu$		$\hat{\tau}_\mu$		$\hat{\rho}$		$\hat{\tau}$		$\hat{\rho}_\mu$		$\hat{\tau}_\mu$	
$\phi \backslash m$	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
1.0	.05	.04	.04	.04	.04	.04	.04	.03	.05	.04	.05	.04	.04	.05	.04	.05
.99	.06	.06	.06	.06	.06	.04	.05	.03	.13	.15	.13	.15	.10	.10	.07	.06
.95	.14	.14	.13	.13	.10	.09	.06	.06	.90	.88	.90	.87	.63	.61	.43	.41
.90	.33	.32	.32	.30	.18	.17	.10	.09	1.0	.99	1.0	.99	.99	.99	.96	.95

	$n \times m = 100$								$n \times m = 400$							
	$\hat{\rho}$		$\hat{\tau}$		$\hat{\rho}_\mu$		$\hat{\tau}_\mu$		$\hat{\rho}$		$\hat{\tau}$		$\hat{\rho}_\mu$		$\hat{\tau}_\mu$	
$\phi \backslash m$	1	4	1	4	1	4	1	4	1	4	1	4	1	4	1	4
1.0	.05	.05	.05	.04	.04	.04	.04	.03	.04	.05	.04	.04	.05	.04	.04	.04
.99	.07	.08	.06	.07	.07	.05	.06	.03	.22	.22	.22	.22	.14	.13	.08	.08
.95	.31	.30	.31	.28	.18	.15	.11	.08	.99	.99	.99	.99	.94	.92	.83	.80
.90	.75	.69	.75	.67	.46	.37	.31	.022	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.99

	$n \times m = 175$								$n \times m = 350$							
	$\hat{\rho}$		$\hat{\tau}$		$\hat{\rho}_\mu$		$\hat{\tau}_\mu$		$\hat{\rho}$		$\hat{\tau}$		$\hat{\rho}_\mu$		$\hat{\tau}_\mu$	
$\phi \backslash m$	1	7	1	7	1	7	1	7	1	7	1	7	1	7	1	7
1.0	.05	.04	.04	.03	.04	.03	.04	.02	.04	.05	.04	.05	.05	.04	.05	.03
.99	.11	.10	.11	.09	.09	.06	.06	.04	.20	.19	.20	.18	.13	.12	.08	.06
.95	.66	.60	.66	.57	.39	.31	.24	.17	.99	.97	.99	.97	.88	.79	.72	.61
.90	.99	.95	.99	.93	.88	.72	.73	.51	1.0	1.0	1.0	.99	1.0	.99	.99	.98

	$n \times m = 300$								$n \times m = 600$							
	$\hat{\rho}$		$\hat{\tau}$		$\hat{\rho}_\mu$		$\hat{\tau}_\mu$		$\hat{\rho}$		$\hat{\tau}$		$\hat{\rho}_\mu$		$\hat{\tau}_\mu$	
$\phi \backslash m$	1	12	1	12	1	12	1	12	1	12	1	12	1	12	1	12
1.0	.05	.05	.04	.04	.04	.03	.05	.02	.04	.05	.04	.04	.05	.04	.04	.03
.99	.17	.17	.17	.15	.12	.10	.07	.05	.41	.39	.40	.38	.23	.22	.13	.13
.95	.97	.89	.96	.87	.77	.61	.58	.40	1.0	1.0	1.0	.99	1.0	.99	.99	.95
.90	1.0	.99	1.0	.99	.99	.94	.99	.85	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

APPENDIX — PROOF OF THEOREM 1

Proof of Theorem 1-i). We first define some notations. Let $Y_k = (y_{1-k}, y_{2-k}, ..., y_{n-k})'$, $Z_k = Y_k - Y_{k-1}$, $e_k(\rho) = (e_{1-k}(\rho), ..., e_{n-k}(\rho))'$, $e_{\rho k}(\rho) = \partial\ e_k(\rho)/\ \partial\rho$, $e_{\rho\rho k}(\rho) = \partial^2 e_k(\rho)/\ \partial\rho^2$, $k = 0, 1, 2, ...$, where $y_i = e_i(\rho) = e_{\rho i}(\rho) = e_{\rho\rho i}(\rho) = 0$ for $i \le 0$. Also let $\bar{e} = e(\bar{\rho})$, $\bar{e}_\rho = e_\rho(\bar{\rho})$, $\bar{e}_{\rho\rho} = e_{\rho\rho}(\bar{\rho})$. Letting ρ^* and $\hat{\rho}$ be given, define e^*, e^*_ρ , $e^*_{\rho\rho}$, $\hat{e}$, $\hat{e}_\rho$, $\hat{e}_{\rho\rho}$ similarly. Note that the transformation (3.3) of $(Y_0 - \rho\ Y_1)$ to $e_0(\rho)$ is a linear transformation. Let $D = D(\beta)$ be $(n \times n)$ matrix which define this transformation. Then

$$e_0(\rho) = D(Y_0 - \rho\ Y_1) = D\ Z_0 + (1-\rho)D\ Y_1. \tag{A.1}$$

It is easy to show that D is a lower triangular matrix with $(i,j)-th$ element $(-\beta)^{i-j}$ for $i \ge j$. We need the following Lemma A.1.

Lemma A.1. If $(\rho^* - 1) = o_p(n^{-1/2})$ then we have $e^{*\prime}_\rho e^*_\rho = O_p(n^2)$, $(e^{*\prime}_\rho e^*_\rho)^{-1} = O_p(n^{-2})$, $e^{*\prime}_\rho e^* = O_p(n^{3/2})$, $e^{*\prime}_{\rho\rho} e^* = O_p(n^{3/2})$, and $e^{*\prime}_{\rho\rho} e^*_\rho = O_p(n^2)$.

Proof of Lemma A.1. From (3.3)-(3.4),

$$e^* = D^* Z_0 + (1-\rho^*)D^* Y_1, \tag{A.2}$$
$$e^*_\rho = -D^*(Y_1 + \beta^*_\rho e^*_1), \tag{A.3}$$
$$e^*_{\rho\rho} = -D^*(\beta^*_{\rho\rho} e^*_1 + 2\beta^*_\rho e^*_{\rho 1}), \tag{A.4}$$

where $D^* = D(\beta^*)$, $\beta^* = \beta(\rho *)$, $\beta^*_\rho = \beta_\rho(\rho^*)$, $\beta^*_{\rho\rho} = \beta_{\rho\rho}(\rho^*)$, $\beta_{\rho\rho} = d^2\beta\ /\ d\rho^2$. Shin and Fuller (1992, Lemma 3.3) showed that $\sup_{n,\beta} \|D'(\beta)D(\beta)\| = O(1)$, where $\|A\|$ denotes the supremum norm of matrix A. Therefore, $\sup_\beta \|D(\beta)Z_0\| = O_p(n^{1/2})$ because $\|Z_0\| = O_p(n^{1/2})$. Also $\sup_\beta \|D(\beta)Y_0\| = O_p(n)$ because $\|Y_0\| = O_p(n)$ by Lemma 4.7 of Shin and Fuller. The notation $\|h\|$ denotes the Euclidean norm of vector h. They(Lemma 4.6, p. 25) also showed that $\sup_\beta |Y_1'D'(\beta)D(\beta)Z_0| = O_p(n)$ and $\inf_\beta |Y_1'D'(\beta)D(\beta)Y_1|^{-1} = O_p(n^{-2})$. Therefore, Lemma A.1.follows.

We now prove Theorem 1-i). Let $\bar{c} = \bar{e}'_\rho \bar{e}_\rho$ and $\bar{d} = \bar{e}'_\rho \bar{e}$. Applying Taylor expansion

$$\begin{aligned} n(\hat{\rho}-1) &= n(\bar{\rho}-1) - n\bar{d}/\bar{c} \\ &= -n\ e^{o\prime}_\rho e^o/\ e^{o\prime}_\rho e^o_\rho + n(\bar{\rho}-1)[(c^* - d^*_\rho)/c^* + d^* c^*_\rho/\ c^{*2}], \end{aligned}$$

where $e^o = e(1)$, $e^o_\rho = e_\rho(1)$, $c^* = e^{*\prime}_\rho e^*_\rho$, $d^* = e^{*\prime}_\rho e^*$, $c^*_\rho = 2e^{*\prime}_\rho e^*_{\rho\rho}$, $d^*_\rho = e^{*\prime}_\rho e^*_\rho + e^{*\prime}_{\rho\rho} e^*$. By Shin and Fuller(1992, p.33),

$$\begin{aligned} (n^{-1}e^{o\prime}_\rho e^o, n^{-2}e^{o\prime}_\rho e^o_\rho) &= (-n^{-1}Y_1'D^{o\prime}D^o Z_0, n^{-2}Y_1'D^{o\prime}D^o Y_1) + O_p(n^{-1/2}) \\ &= (n^{-1}\sum_{j=2}^{n}\sum_{i=1}^{j-1} e_i e_j, n^{-2}\sum_{j=2}^{n}(\sum_{i=1}^{j-1} e_i)^2) + O_p(n^{-1/2}), \end{aligned}$$

where $D^o = D(\beta(1))$. Hence $(n^{-1}e_{\rho}^{o\prime}e^o,\ n^{-2}e_{\rho}^{o\prime}e_{\rho}^o) \rightarrow_d \sigma^2\{-2^{-1}(T^2-1),\ G\}$ by Dickey and Fuller(1979). Observe that $(c^* - d_{\rho}^*) = -e_{\rho_\rho}^{*\prime}e^*$. Therefore, by Lemma A.1 and assumption of $(\bar{\rho}-1) = o_p(n^{-1/2})$, the limiting distribution of $n(\hat{\rho}-1)$ is $G^{-1}(T^2-1)/2$. Also by Lemma A.1. together with $(\hat{\rho}-1) = O_p(n^{-1})$ and Shin and Fuller(1992, pp.16-17),

$$s^2 = \hat{e}'\hat{e}/(n-1) = Z_0'D(\beta(\hat{\rho}))'D(\beta(\hat{\rho}))Z_0/(n-1) + O_p(n^{-1}) \rightarrow_p \sigma^2.$$

Also

$$n^{-2}\hat{e}_{\rho}'\hat{e}_{\rho} = n^{-2}e_{\rho}^{o\prime}e_{\rho}^o + O_p(n^{-1/2}) \rightarrow_d \sigma^2 G.$$

Therefore, the limiting distribution of $\hat{\tau}$ follows.

Proof of Theorem 1-ii). Define $e_k(\rho,\ \mu) = (e_{1-k}(\rho,\ \mu),\ \ldots,\ e_{n-k}(\rho,\ \mu))'$ and define Y_k, Z_k, $e_{\rho k}$, $e_{\mu k}$, $e_{\rho\rho k}$, $e_{\rho\mu k}$, $e_{\mu\mu k}$, $e_{\rho k}^o$, $e_{\mu k}^o$, $e_{\rho\rho k}^o$, $e_{\rho\mu k}^o$, $e_{\mu\mu k}^o$, $e_{\rho k}^*$, $e_{\mu k}^*$, $e_{\rho\rho k}^*$, $e_{\rho\mu k}^*$, $e_{\mu\mu k}^*$, $\bar{e}_{\rho k}$, $\bar{e}_{\mu k}$, $\bar{e}_{\rho\rho k}$, $\bar{e}_{\rho\mu k}$, $\bar{e}_{\mu\mu k}$, $\hat{e}_{\rho k}$, $\hat{e}_{\mu k}$, $\hat{e}_{\rho\rho k}$, $\hat{e}_{\rho\mu k}$, $\hat{e}_{\mu\mu k}$, D, and D^o similarly as in the proof of Theorem 1-i). Applying Taylor expansion,

$$\begin{aligned} n(\hat{\rho}_\mu - 1) &= n(\bar{\rho}_\mu - 1) - n\bar{a}/\bar{b} \\ &= -n\ a^o/\ b^o + n(\bar{\rho}_\mu - 1)[(b^* - a_{\rho}^*)/b^* + a^*b_{\rho}^*/(b^*)^2] - n\bar{\mu}(a_{\mu}^*b^* - a^*b_{\mu}^*)/(b^*)^2, \end{aligned}$$

where a_ρ, a_μ, b_ρ, b_μ are derivatives of a and b. Also $(a^o,\ b^o)$ is evaluated at $(1,\ 0)$ and $(a^*,\ b^*,\ a_{\rho}^*\ ,\ b_{\rho}^*,\ a_{\mu}^*,\ b_{\mu}^*)$ is evaluated at $(\rho_{\mu}^*\ ,\ \mu^*)$ which is between $(1,\ 0)$ and $(\bar{\rho}_\mu\ ,\ \bar{\mu})$. By Theorem 5.6 of Shin(1990),

$$\begin{aligned} n^{-1}e_{\mu}^{o\prime}e_{\mu}^o &= n^{-1}1'D^{o\prime}D^o 1 = (1-\beta)^{-2} + O_p(n^{-1}), \qquad \text{(A.5)} \\ n^{-3/2}e_{\mu}^{o\prime}e_{\rho}^o &= n^{-3/2}1'D^{o\prime}D^oY_1 + O_p(n^{-1/2}) \\ &= (1-\beta)^{-1}n^{-3/2}\textstyle\sum_{j=2}^{n}\sum_{i=1}^{j-1} e_i + O_p(n^{-1/2}), \\ n^{-1/2}e_{\mu}^{o\prime}e^o &= n^{-1/2}1'D^{o\prime}D\ Z_0 = (1-\beta)^{-1}\sum_{2}^{n} e_j + O_p(n^{-1/2}), \end{aligned}$$

where $1 = (1,\ 1\ \ldots,\ 1)'$. Therefore, together with (A.5) and Dickey and Fuller(1979),

$$\begin{aligned} n\ b^o/\ a^o &= [e_{\mu}^{o\prime}e_{\mu}^o e_{\rho}^{o\prime}e^o - e_{\rho}^{o\prime}e_{\mu}^o e_{\mu}^{o\prime}e^o]/[e_{\rho}^{o\prime}e_{\rho}^o e_{\mu}^{o\prime}e_{\mu}^o - e_{\rho}^{o\prime}e_{\mu}^o e_{\rho}^{o\prime}e_{\mu}^o] \\ &\rightarrow_d 2^{-1}(T^2 - 1 - 2TW)/(G - W^2). \end{aligned}$$

Now, by Theorem 5.6 of Shin(1990) together with $(\bar{\rho}-1) = o_p(n^{-1/2})$, $\bar{\mu} = o_p(1)$ and following the method of Lemma A.1, we have $e^{*\prime}e_{\rho}^* = O_p(n^{3/2})$, $e^{*\prime}e_{\mu}^* = O_p(n)$, $e^{*\prime}e_{\rho\rho}^* = O_p(n^{3/2})$, $e^{*\prime}e_{\rho\mu}^* = O_p(n)$, $e_{\rho}^{*\prime}e_{\rho}^* = O_p(n^2)$, $e_{\rho}^{*\prime}e_{\mu}^* = O_p(n^{3/2})$, $e_{\rho}^{*\prime}e_{\rho\rho}^* = O_p(n^2)$, $e_{\rho}^{*\prime}e_{\rho\mu}^* = O_p(n^{3/2})$, $e_{\rho}^{*\prime}e_{\mu\mu}^* = 0$, $e_{\mu}^{*\prime}e_{\mu}^* = O_p(n)$, $e_{\mu}^{*\prime}e_{\rho\rho}^* = O_p(n^{3/2})$, $e_{\mu}^{*\prime}e_{\rho\mu}^* = O_p(n)$,

$e_\mu^{*\prime} e_{\mu\mu}^* = 0$. Hence, $b^* = O_p(n^{5/2})$, $a_\rho^* = O_p(n^3)$, $a_\mu^* = O_p(n^3)$, $b_\rho^* - a^* = o_p(n^{5/2})$, $b_\mu^* = o_p(n^2)$. Also $(a^*)^{-1} = O_p(n^{-3})$ because $n^{-3}a^* \rightarrow_d \sigma^2(1-\beta)^{-2}(G - W^2)$ and $(G - W^2) > 0$ *a.s.* Therefore,

$$n(\hat{\rho}_\mu - 1) = -n\, b^o/\, a^o + o_p(1) \rightarrow_d 2^{-1}(G - W^2)^{-1}(T^2 - 1 - 2TW).$$

Finally, we derive the limiting distribution of $\hat{\tau}_\mu$. We have

$$n^2 c_{11} = n^{-1}\hat{e}_\mu'\hat{e}_\mu/[n^{-3}\{\hat{e}_\rho'\hat{e}_\rho\hat{e_\mu}'\hat{e}_\mu - (\hat{e}_\rho'\hat{e}_\mu)^2\}] \rightarrow_d \sigma^{-2}(G - W^2)^{-1}$$

because

$$n^{-3}\{\hat{e}_\rho'\hat{e}_\rho\hat{e_\mu}'\hat{e}_\mu - (\hat{e}_\rho'\hat{e}_\mu)^2\}] \rightarrow_d \sigma^2(1-\beta)^{-2}(G - W^2)$$

and

$$n^{-1}\hat{e}_\mu'\hat{e}_\mu \rightarrow_d (1-\beta)^{-2}.$$

Also $n^{-1}\hat{e}'\hat{e} \rightarrow_d \sigma^2$. Therefore,

$$\hat{\tau}_\mu = n(\hat{\rho}_\mu - 1)/(s_\mu^2/n^2 c_{11})^{1/2} \rightarrow_d 2^{-1}(G^{-1} - W^2)^{-1/2}(T^2 - 1 - 2TW).$$

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