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A Laplacian Autoregressive Moving-Average Time Series Model

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ABSTRACT

A moving average model, LMA(q) and an autoregressive-moving average model, NLARMA(p,q), with Laplacian marginal distribution are constructed and their properties are discussed; Their autocorrelation structures are completely analogous to those of Gaussian process and they are partially time reversible in the third order moments. Finally, we study the mixing property of NLARMA process.

KEYWORDS: Laplacian Moving Average(LMA) process, New Laplacian Autoregressive Moving Average(NLARMA) process, autocorrelation structure, partially time reversibility, strong-mixing property

1. INTRODUCTION

A study on non-Gaussian time series models has been continued in the past two decades. To describe a Laplacian process with Laplacian marginal distribution, Dewald and Lewis(1985) introduced the New Laplacian Autoregressive(NLAR) process which followed the earlier work by Lawrance and Lewis(1981, 1985), the New Exponential Autoregressive(NEAR) process. The motivation behind Laplacian process is in the need of models for correlated random variables with a larger kurtosis or a heavier tails than is exhibited by Gaussian variates. Son and Cho(1988) developed the modelling and forecasting procedures in the NLAR process. In this paper we

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discuss the Laplacian Moving Average(LMA) process and the New Laplacian Autoregressive Moving Average (NLARMA) process in relation to the NLAR process.

In section 2 and section 3, we construct the LMA(q) model and the NLARMA(p,q) model, respectively. In secton 4, the autocorrelation structures are obtained. *Partially time reversibility* in the third order moments is discussed in section 5. Finally, we show the mixing property of NLARMA process.

2. LAPLACIAN MOVING AVERAGE PROCESS

Throughout this paper we assume without loss of generality that the marginal distribution of $\{X_t\}$ is the standard Laplacian distribution for simplicity. The EMA(1) process developed by Lawrance and Lewis(1977) directly leads to the construction of a first-order Laplacian Moving Average(LMA(1)) process,

$$X_t = \begin{cases} \theta E_t & \text{w.p. } \theta^2 \\ E_{t-1} + \theta E_t & \text{w.p. } 1-\theta^2, \end{cases} \tag{2.1}$$

for $t = 0, \pm 1, \pm 2, \ldots$, and $|\theta| < 1$, where $\{E_t\}$ is a sequence of i.i.d. standard Laplacian variables. It is easily shown that the $\{X_t\}$ has a standard Laplacian marginal distribution following the relation

$$\begin{aligned} M_X(s) &= \theta^2 M_E(s\theta) + (1-\theta^2) M_E(s) M_E(s\theta) \\ &= \frac{\theta^2}{1-s^2\theta^2} + \frac{1-\theta^2}{(1-s^2)(1-s^2\theta^2)} \\ &= \frac{1}{1-s^2}, \end{aligned} \tag{2.2}$$

where $M_X(s)$ and $M_E(s)$ are the moment generating function of X_t and E_t, respectively.

As an extension to the second-order moving average process, we replace E_{t-1} in (2.1) by another LMA(1) variable, a random linear combination of $\theta_1 E_{t-1}$ and $\theta_1 E_{t-1} + E_{t-2}$, which are still Laplacian and independent of E_t. Thus, the LMA(2) process is given by

$$X_t = \begin{cases} \theta_0 E_t & \text{w.p. } \theta_0^2 \\ \theta_1 E_{t-1} + \theta_0 E_t & \text{w.p. } (1-\theta_0^2)\theta_1^2 \\ E_{t-2} + \theta_1 E_{t-1} + \theta_0 E_t & \text{w.p. } (1-\theta_0^2)(1-\theta_1^2), \end{cases} \tag{2.3}$$

for $t = 0, \pm 1, \pm 2, \ldots$ and $|\theta_i| < 1$, $i = 0, 1$.

The general LMA(q) process is constructed

$$X_t = \begin{cases} \theta_0 E_t & \text{w.p.} \quad b_0 \\ \theta_1 E_{t-1} + \theta_0 E_t & \text{w.p.} \quad b_1 \\ \vdots & \quad \vdots \\ \theta_{q-1} E_{t-q+1} + \cdots + \theta_1 E_{t-1} + \theta_0 E_t & \text{w.p.} \quad b_{q-1} \\ E_{t-q} + \theta_{q-1} E_{t-q+1} + \cdots + \theta_1 E_{t-1} + \theta_0 E_t & \text{w.p.} \quad b_q, \end{cases} \tag{2.4}$$

for $t = 0, \pm 1, \pm 2, \ldots$ and $|\theta_i| < 1$, $i = 0, 1, 2, \cdots, q-1$, where

$$b_i = \begin{cases} \theta_0^2 & , i = 0 \\ (1-\theta_0^2)\cdots(1-\theta_{i-1}^2)\theta_i^2 & , 1 \le i \le q-1, \ (q \ge 2) \\ (1-\theta_0^2)\cdots(1-\theta_{q-1}^2) & , i = q. \end{cases} \tag{2.5}$$

3. NEW LAPLACIAN AUTOREGRESSIVE MOVING AVERAGE PROCESS

We have constructed the moving average process in Laplacian variables. Following Lawrance and Lewis(1980) the NLARMA(p,q) process can be constructed by replacing an E_{t-q} variable in the LMA(q) process of (2.4) by an NLAR(p) variable which is independent of $E_t, E_{t-1}, \cdots, E_{t-q+1}$. Thus the NLARMA(p,q) process is defined,

$$X_t = \begin{cases} \theta_0 E_t & \text{w.p.} \quad b_0 \\ \theta_1 E_{t-1} + \theta_0 E_t & \text{w.p.} \quad b_1 \\ \vdots & \quad \vdots \\ \theta_{q-1} E_{t-q+1} + \cdots + \theta_1 E_{t-1} + \theta_0 E_t & \text{w.p.} \quad b_{q-1} \\ A_{t-q} + \theta_{q-1} E_{t-q+1} + \cdots + \theta_1 E_{t-1} + \theta_0 E_t & \text{w.p.} \quad b_q, \end{cases} \tag{3.1}$$

for $t = 0, \pm 1, \pm 2, \ldots$ and $|\theta_i| < 1$, $i = 0, 1, 2, \cdots, q-1$, where b_i's are defined in (2.5), $\{E_t\}$ is a sequence of i.i.d. standard Laplacian variables, and

$$A_{t-q} = \left\{ \begin{array}{lll} \phi A_{t-q-1} & \text{w.p.} & a_1 \\ \phi A_{t-q-2} & \text{w.p.} & a_2 \\ \vdots & & \vdots \\ \phi A_{t-q-p} & \text{w.p.} & a_p \\ 0 & \text{w.p.} & 1-p^* \end{array} \right\} + \varepsilon_{t-q} \tag{3.2}$$

with $p^* = \sum_{i=1}^{p} a_i$ and

$$\varepsilon_{t-q} = \begin{cases} E_{t-q} & \text{w.p. } 1-\pi \\ \sqrt{1-p^*}|\phi| E_{t-q} & \text{w.p. } \pi = \phi^2 p^* / (1-(1-p^*)\phi^2). \end{cases} \tag{3.3}$$

for $0<|\phi|<1$ and $0<p<1$.

The NLARMA(1,1) process is the simplest form of NLARMA(p,q) process;

$$X_t = \begin{cases} \theta E_t & \text{w.p. } \theta^2 \\ A_{t-1}+\theta E_t & \text{w.p. } 1-\theta^2, \end{cases} \tag{3.4}$$

for $t=0,\pm1,\pm2,\ldots$ and $|\theta|<1$, where

$$A_{t-1} = \left\{ \begin{array}{ll} \phi A_{t-2} & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{array} \right\} + \varepsilon_{t-1} \tag{3.5}$$

and

$$\varepsilon_{t-1} = \begin{cases} E_{t-1} & \text{w.p. } 1-\pi \\ \sqrt{1-p}\,|\phi| E_{t-1} & \text{w.p. } \pi = \phi^2 p/(1-(1-p)\phi^2), \end{cases} \tag{3.6}$$

for $0<|\phi|<1$ and $0<p<1$. We can easily show that the marginal distribution of $\{X_t\}$ in (3.4) is a standard Laplacian distribution as follows;

$$\begin{aligned} M_X(s) &= \theta^2 M_E(s\theta) + (1-\theta^2) M_E(s\theta) M_A(s) \\ &= \frac{\theta^2}{1-s^2\theta^2} + \frac{1-\theta^2}{(1-s^2\theta^2)(1-s^2)} \\ &= \frac{1}{1-s^2}. \end{aligned} \tag{3.7}$$

4. AUTOCORRELATION STRUCTURE

The autocorrelation function(ACF) of LMA(q) process is calculated as follows;

$$\begin{aligned} \mathrm{E}(X_t X_{t-k}) &= \sum_{i=0}^{q} \theta_i \sum_{j=i}^{q} b_j \mathrm{E}(E_{t-i} X_{t-k}) \\ &= \sum_{i=0}^{q} \frac{b_i}{\theta_i} \mathrm{E}(E_{t-i} X_{t-k}), \end{aligned} \tag{4.1}$$

since $b_i = \theta_i^2(b_i + b_{i+1} + \cdots + b_q) = \theta_i^2 \sum_{j=i}^{q} b_j$ and $\theta_i \sum_{j=i}^{q} b_j = b_i/\theta_i$ when $\theta_i \neq 0$. Accordingly we have

$$\mathrm{Cov}(X_t, X_{t-k}) = \sum_{i=0}^{q} \frac{b_i}{\theta_i} \mathrm{Cov}(E_{t-i} X_{t-k}). \tag{4.2}$$

Since

$$\operatorname{Cov}(E_{t-i}, X_{t-k}) = \begin{cases} 2\theta_{i-k} \sum_{j=i-k}^{q} b_j & , k \le i \le k+q \\ 0 & , \text{otherwise,} \end{cases} \tag{4.3}$$

using (2.4), it follows that

$$\operatorname{Cov}(X_t, X_{t-k}) = 2 \sum_{i=0}^{q-k} \frac{b_i b_{i+k}}{\theta_i \theta_{i+k}}, \tag{4.4}$$

which gives the ACF,

$$\rho_k = \begin{cases} \sum_{i=0}^{q-k} c_i c_{i+k} & , 1 \le k \le q \\ 0 & , k > q, \end{cases} \tag{4.5}$$

where $c_i = b_i/\theta_i$ if $\theta_i \neq 0$ and $c_i = 0$ if $\theta_i = 0$. The result (4.5) implies that the ACF of LMA(q) clearly has the cut-off property, which is completely analogous to that of Gaussian MA(q) process. For example, the ACF of LMA(1) of (2.1) is

$$\rho_1 = \theta(1-\theta^2) \text{ and } \rho_k = 0, \quad k \ge 2. \tag{4.6}$$

The theoretically admissible range of the ACF of LMA(1) is [-0.385, 0.385] , while [-0.5, 0.5] for Gaussian MA(1) process. The ACF of LMA(2) process of (2.3) is

$$\begin{cases} \rho_1 &= \theta_1(1-\theta_0^2)\{\theta_0 + (1-\theta_0^2)(1-\theta_1^2)\} \\ \rho_2 &= \theta_0(1-\theta_0^2)(1-\theta_1^2) \\ \rho_k &= 0, \ k \ge 3. \end{cases} \tag{4.7}$$

The admissible region for ρ_1 and ρ_2 in LMA(2) shrinks to the interior of that of Gaussian MA(2).

Next, we derive the autocorreleation function of the NLARMA(p,q) process of (3.1). Its initial step gives

$$\mathrm{E}(X_t X_{t-k}) = \sum_{i=0}^{q-1} \frac{b_i}{\theta_i} \mathrm{E}(E_{t-i} X_{t-k}) + b_q \mathrm{E}(A_{t-q} X_{t-k}), \ k \ge 1, \tag{4.8}$$

noting that $\theta_{q-i} \sum_{j=0}^{i} b_{q-j} = b_{q-i}/\theta_{q-i}$ for $\theta_{q-i} \neq 0$ and $0 \le i \le q-1$. Letting $\rho_k = Corr(X_t, X_{t-k})$ and $R_k = Corr(E_t, X_{t-k})$ for $k \ge 1$, (4.8) becomes

$$\rho_k = \sum_{i=0}^{q-1} \frac{b_i}{\theta_i} R_{k-i} + b_q Corr(A_{t-q}, X_{t-k}), \ k \ge 1. \tag{4.9}$$

From

$$\mathrm{E}(A_{t-q}X_{t-k}) = \phi \sum_{i=1}^{p} a_i \mathrm{E}(A_{t-q-i}X_{t-k}) + \mathrm{E}(\varepsilon_{t-q}X_{t-k}) \tag{4.10}$$

and

$$Var(A_{t-q}) = (1-\phi^2 p^*)^{-1} Var(\varepsilon_{t-q}), \tag{4.11}$$

we have

$$\begin{aligned} &Corr(A_{t-q}, X_{t-k}) \\ &= \phi \sum_{i=1}^{p} a_i Corr(A_{t-q-i}, X_{t-k}) + \sqrt{1-\phi^2 p^*} Corr(\varepsilon_{t-q}, X_{t-k}), \ k \geq 1. \end{aligned} \tag{4.12}$$

In case of $k = 0$, we do not have (4.8). Thus if $i = k \leq p$ when $i = k$, $Corr(A_{t-q}, X_t) = b_q$. Now, from (4.9) and (4.12), when $1 \leq k \leq p$

$$\begin{aligned} \rho_k &= \sum_{i=0}^{q-1} \frac{b_i}{\theta_i} R_{k-i} + b_q \phi \sum_{i=1}^{p} a_i Corr(A_{t-q-i} X_{t-k}) \\ &\quad + b_q \sqrt{1-\phi^2 p^*} Corr(\varepsilon_{t-q} X_{t-k}) \\ &= \sum_{i=0}^{q-1} \frac{b_i}{\theta_i} R_{k-i} + \phi \sum_{i=1, i\neq k}^{p} a_i (\rho_{k-i} - \sum_{j=0}^{q-1} \frac{b_j}{\theta_j} R_{k-j-i}) \\ &\quad + b_q^2 \phi a_k + b_q \sqrt{1-\phi^2 p^*} Corr(\varepsilon_{t-q}, X_{t-k}) \\ &= \phi \sum_{i=1}^{p} a_i \rho_{k-i} + \sum_{i=0}^{q-1} \frac{b_i}{\theta_i} (R_{k-i} - \phi \sum_{j=1, j\neq k}^{p} a_j R_{k-j-i}) \\ &\quad + b_q \sqrt{1-\phi^2 p^*} Corr(\varepsilon_{t-q}, X_{t-k}), \end{aligned} \tag{4.13}$$

where $\rho_0 \equiv b_q^2$ and

$$Corr(\varepsilon_{t-q}, X_{t-k}) = \frac{(1-\pi+\pi|\phi|\sqrt{1-p^*})\theta_{q-k} \sum_{i=q-k}^{q} b_i}{\sqrt{1-\pi+\pi(1-p^*)\phi^2}}. \tag{4.14}$$

The final result of ρ_k when $k > p$ is the same as (4.13) only if the term $\sum_{j=1, j\neq k}^{p} a_j R_{k-j-i}$ is replaced by $\sum_{j=1}^{p} a_j R_{k-j-i}$. Since $R_{k-i} = Corr(E_{t-i}, X_{t-k}) = 0$ for $k > i$, for $k \geq p+q$, (4.13) gives

$$\rho_k = \phi \sum_{i=1}^{p} a_i \rho_{k-i}, \tag{4.15}$$

which is the same as the NLAR(p) process,

$$\rho_k = \phi \sum_{i=1}^{p} a_i \rho_{k-i} - \phi \sum_{i=0}^{q-1} \frac{b_i}{\theta_i} \sum_{j=1, j \neq k}^{p} a_j R_{k-j-i}, \; q < k < p+q, \tag{4.16}$$

and finally the equation (4.13) for ρ_k holds when $1 \leq k \leq q$.

For example, we can obtain the ACF of NLARMA(1,1) process as follows

$$\begin{aligned} \rho_1 &= \phi p (1-\theta^2)^2 + \frac{(1-\theta^2)(1-\phi^2 p)(1-\pi+\pi|\phi|\sqrt{1-p})\theta}{\sqrt{1-\pi+\pi(1-p)\phi^2}}, \\ \rho_k &= (\phi p)^{k-1} \rho_1, \; k \geq 2. \end{aligned} \tag{4.17}$$

The ACF of NLARMA(1,1), in case of $\phi = 0$, is the same as that of LMA(1) and the ACF of NLARMA(1,1), in case of $\theta = 0$, is the same as that of NLAR(1).The admissible region for ρ_1 and ρ_2 in NLARMA(1,1) shrinks to the interior of that in Gaussian ARMA(1,1).

5. PARTIALLY TIME REVERSIBILITY

The moment generating function of joint p.d.f. of X_t and X_{t-1} of LMA(1) of (2.1) is

$$\begin{aligned} M_{X_t, X_{t-1}}(s_1, s_2) &= \mathrm{E}\{e^{s_1 X_t + s_2 X_{t-1}}\} \\ &= (\theta^2)^2 \mathrm{E}\{e^{s_1 \theta E_t + s_2 \theta E_{t-1}}\} \\ &\quad + \theta^2(1-\theta^2)\mathrm{E}\{e^{s_1 \theta E_t + s_2(\theta E_{t-1} + E_{t-2})}\} \\ &\quad + (1-\theta^2)\theta^2 \mathrm{E}\{e^{s_1(\theta E_t + E_{t-1}) + s_2 \theta E_{t-1}}\} \\ &\quad + (1-\theta^2)^2 \mathrm{E}\{e^{s_1(\theta E_t + E_{t-1}) + s_2(\theta E_{t-1} + E_{t-2})}\} \\ &= \frac{1-\theta^2(s_1^2+s_2^2+2s_1 s_2 \theta)}{(1-s_2^2)(1-s_1^2\theta^2)\{1-(s_1+s_2\theta)^2\}}. \end{aligned} \tag{5.1}$$

We note that (5.1) is not symmetric in s_1 and s_2, and so the LMA(1) process is not *time reversible*.

But

$$E(X_t^2 X_{t-l}) = E(X_t X_{t-l}^2) = 0 \;\; \text{for all } l \geq 0, \tag{5.2}$$

which implies that the LMA(1) process is *partially time reversible* in the sense of third order moments. To show the *partially time reversibility* of LMA(1) we rewrite the LMA(1) of (2.1) as follows

$$X_t = \theta E_t + W_t E_{t-1}, \tag{5.3}$$

where $\{W_t\}$ is a sequence of i.i.d. Bernoulli$(1-\theta^2)$ r.v's and independent of $\{E_t\}$ for all t. Thus we have following two equations,

$$\begin{aligned} E(X_t^2 X_{t-l}) &= E\{(\theta E_t + W_t E_{t-1})^2 X_{t-l}\} \\ &= E\{\theta^2 E_t^2 X_{t-l} + W_t^2 E_{t-1}^2 X_{t-l} + 2\theta W_t E_t E_{t-1} X_{t-l}\} \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} E(X_t X_{t-l}^2) &= E\{(\theta E_t + W_t E_{t-1}) X_{t-l}^2\} \\ &= E\{\theta E_t X_{t-l}^2 + W_t E_{t-1} X_{t-l}^2\}. \end{aligned} \tag{5.5}$$

Solving (5.4) and (5.5) for all $l \geq 1$ yields $E(X_t^2 X_{t-l}) = E(X_t X_{t-l}^2) = 0$ for all $l \geq 1$. Also, We know that $E(X_t^3) = 0$ for all t since X_t is a marginally standard Laplacian variable. Hence, the equation (5.2) holds.

From the structure of (5.1) we expect that the joint distribution of (X_t, X_{t-1}) be a mixture of joint distributions of $(\theta E_t, \theta E_{t-1})$, $(\theta E_t, \theta E_{t-1} + E_{t-2})$, $(\theta E_t + E_{t-1}, \theta E_{t-1})$, and $(\theta E_t + E_{t-1}, \theta E_{t-1} + E_{t-2})$ with the corresponding probabilities θ^4, $\theta^2(1-\theta^2)$, $\theta^2(1-\theta^2)$, and $(1-\theta^2)^2$, respectively.

Now, we can easily know that NLARMA(1,1) process, a mixture of NLAR(1) and LMA(1), is not *fully time reversible* since neither NLAR(1) process nor LMA(1) process is *fully time reversible*. But by induction, we have $E(X_t^2 X_{t-l}) = E(X_t X_{t-l}^2) = 0$ for all $l \geq 0$, which implies that NLARMA(1,1) is *partially time reversible* in the sense of third order moments. The *partially time reversibility* of NLARMA(1,1) is similarly proved as LMA(1) by using the fact that the NLAR(1) of (3.5) is rewritten by $A_{t-1} = \sum_{j=0}^{\infty} \psi_{j,t-1}\varepsilon_{t-j-1}$, where $\psi_{0,t-1} \equiv 1$ and $\psi_{j,t-1} = \phi^j \prod_{i=0}^{j-1} W_{t-i-1}$, $j = 1, 2, \cdots$.

6. STRONG-MIXING PROPERTY

Son and Cho(1988) showed that the NLAR(1) process is *strong-mixing*. Following them we can easily show the mixing property of $\{X_t\}$ in the NLARMA (1,1). The NLARMA(1,1) in (3.4) can be rewritten as

$$X_{t+k+1} = \begin{cases} \theta E_{t+k+1} & \text{w.p.} \quad \theta^2 \\ \phi^k A_t + \sum_{j=0}^{k-1} \phi^j \varepsilon_{t+k-j} + \theta E_{t+k+1} & \text{w.p.} \quad p^k(1-\theta^2) \\ \sum_{j=0}^{i} \phi^j \varepsilon_{t+k-j} + \theta E_{t+k+1} & \text{w.p.} \quad p^i(1-p)(1-\theta^2), \\ & \qquad i = 0, 1, 2, \ldots, k-1. \end{cases} \tag{6.1}$$

Let $Y = \theta E_{t+k+1} + \sum_{j=0}^{k-1} \phi^j \varepsilon_{t+k-j}$ then from (3.6)

$$M_Y(s) = \mathrm{E}(e^{sY}) = \frac{\phi^{2k} + (1-\phi^{2k})/(1-s^2)}{(1-\theta^2 s^2)\prod_{j=1}^{k}\{1-\phi^{2j}(1-p)s^2\}}. \tag{6.2}$$

The moment generating function (6.2) of the distribution of Y gives the following relation,

$$Y = \theta L_{k+1} \sum_{j=1}^{k} |\phi|^j \sqrt{1-p} L_j + \begin{cases} 0 & \text{w.p. } \phi^{2k} \\ L_0 & \text{w.p. } 1-\phi^{2k}, \end{cases} \tag{6.3}$$

where $\{L_j : j = 0, 1, 2, \ldots, k, k+1\}$ is a sequence of i.i.d. standard Laplacian variables. Now, following Son and Cho(1988), we have

$$dF_Y(y) \le C e^{-|y|}\, dy, \tag{6.4}$$

where C is a constant.

Suppose F_{t+1} and G_{t+k+1} are sigma fields generated by $(\ldots, A_{t-1}, A_t, E_1, E_2, \ldots, E_{t+1})$ and $(A_{t+k}, A_{t+k+1}, \ldots, E_{t+k+1}, E_{t+k+2}, \ldots)$, respectively. Let $\mathcal{L}^2(F_{t+1})$ and $\mathcal{L}^2(G_{t+k+1})$ be the collections of real valued functions that are measurable with respect to F_{t+1} and G_{t+k+1}, respectively, and $f \in \mathcal{L}^2(F_{t+1})$ and $g \in \mathcal{L}^2(G_{t+k+1})$ in the sense that $\mathrm{E}(f^2) < \infty$ and $\mathrm{E}(g^2) < \infty$. Consider the events $Q = \{X_{t+k+1} = \theta E_{t+k+1}\}$ with $\Pr(Q) = \theta^2$, $R = \{X_{t+k+1} = \phi^k A_t + \sum_{j=0}^{k-1} \phi^j \varepsilon_{t+k-j} + \theta E_{t+k+1}\}$ with $\Pr(R) = p^k(1-\theta^2)$ and $S_i = \{X_{t+k+1} = \sum_{j=0}^{i} \phi^j \varepsilon_{t+k-j} + \theta E_{t+k+1}\}$ with $\Pr(S_i) = p^i(1-p)(1-\theta^2)$. for $i = 0, 1, 2, \ldots, k-1$. By the Markov property of $\{A_t\}$ we have that

$$\mathrm{E}(f \cdot g) - \mathrm{E}(f)\mathrm{E}(g) = \mathrm{E}\{f \cdot (\mathrm{E}(g|X_t) - \mathrm{E}(g))\}. \tag{6.5}$$

Now,

$$\begin{aligned} \mathrm{E}(g|A_t) = \;& \theta^2 \mathrm{E}\{(g|A_t)|Q\} + p^k(1-\theta^2)\mathrm{E}\{(g|A_t)|R\} \\ & + \sum_{i=0}^{k-1} p^i(1-p)(1-\theta^2)\mathrm{E}\{(g|A_t)|S_i\} \end{aligned} \tag{6.6}$$

and

$$\mathrm{E}(g) = \theta^2 \mathrm{E}(g|Q) + p^k(1-\theta^2)\mathrm{E}(g|R) + \sum_{i=0}^{k-1} p^i(1-p)(1-\theta^2)\mathrm{E}(g|S_i). \tag{6.7}$$

Since $\mathrm{E}\{(g|A_t)|Q\} = \mathrm{E}(g|Q)$ and $\mathrm{E}\{(g|A_t)|S_i\} = \mathrm{E}(g|S_i)$ for $i = 0, 1, 2, \ldots, k-1$,

$$\mathrm{E}(g|A_t) - \mathrm{E}(g) = p^k(1-\theta^2)\{\mathrm{E}((g|A_t)|R) - \mathrm{E}(g|R)\}. \tag{6.8}$$

$$\begin{aligned} |\mathrm{E}\{(g|A_t)|R\}| &= |\textstyle\int_{-\infty}^{\infty} \mathrm{E}(g|X_{t+k+1}=y)\, dF_Y(y-\phi^k a_t)| \\ &\le C \cdot \textstyle\int_{-\infty}^{\infty} \mathrm{E}(|g||X_{t+k+1}=y) \mathrm{e}^{-|y-\phi^k x_t|}\, dy, \text{ from (6.4)} \\ &\le C \cdot \textstyle\int_{-\infty}^{\infty} \mathrm{E}(|g||X_{t+k+1}=y) \mathrm{e}^{-|y|+\phi^k a_t}\, dy \\ &= 2C \cdot \mathrm{e}^{\phi^k a_t} \mathrm{E}(|g|). \end{aligned} \tag{6.9}$$

And,

$$\begin{aligned} |\mathrm{E}(g|R)| &= |\textstyle\int_{-\infty}^{\infty} \mathrm{E}\{(g|A_t)|R\} \cdot 2^{-1} \cdot \exp(-|a_t|)\, da_t| \\ &\le C \cdot \textstyle\int_{-\infty}^{\infty} \mathrm{e}^{\phi^k a_t} \mathrm{E}(|g|) \mathrm{e}^{-|a_t|}\, da_t, \text{ from (6.9)} \\ &= 2C \cdot \mathrm{E}(|g|)(1-\phi^{2k})^{-1}. \end{aligned} \tag{6.10}$$

After this, following Son and Cho(1988) we can directly derive the fact that NLARMA(1,1) process is *strong-mixing* and asymptotically uncorrelated in the sense of Rosenblatt(1971).

7. FURTHER STUDY

We have derived a moving average model(LMA) with Laplacian marginal distribution. The autoregressive model(NLAR) and the moving average model (LMA) are combined to give a mixed model(NLARMA). They are easily simulated, as is the NLAR process.

About estimation problems, we have failed in writing the likelihood function of LMA and NLARMA. This fact makes impossible to efficiently estimate parameters. The Yule-Walker estimates for θ in the LMA(1) are solutions $\widehat{\theta}$'s which satisfy the equation $\widehat{\theta}^3 - \widehat{\theta} + r_1 = 0$, where $|\widehat{\theta}| < 1$ and $r_1 = \sum_{t=2}^n x_t x_{t-1} / \sum_{t=1}^n x_t^2$. But they are not unique. The NLARMA model is not explicitly estimated by the Yule-Walker estimation because of the overparameterization. But the conditional least square estimation by Klimko and Nelson(1978) can be applied to the estimation of NLARMA model and is under study.

About forecasting problems, the minimum mean square error forecasts can be obtained as is the NLAR process of Son and Cho(1988). Also, the model-free prediction intervals can be obtained using the mixing property.

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