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Nonparametric Estimation of Mean Residual Life Function under Random Censorship†

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ABSTRACT

In the survival analysis the problem of estimating mean residual life function(MRLF) under random censoring is very important. In this paper we propose and study a nonparametric estimator of MRLF, which is a functional form based on the estimator of the survival function due to Susarla and Van Ryzin(1980). The proposed estimator is shown to be better than some other estimators in terms of mean square errors for the exponential and Weibull cases via Monte Carlo simulation studies.

KEYWORDS: Mean Residual Life Function, Random Censorship, Strong Consistency, Weak Convergence, Gaussian Process.

1. INTRODUCTION

Let $X_1, \cdots, X_n$ be independent and identically distributed(i.i.d.) random survival times with common continuous distribution $F(x)$ on $[0, \infty)$ with $F(0) = 0$ and a finite mean μ . The MRLF at age $x(\geq 0)$ is defined as

$$e(x) = \begin{cases} E[X - x | X > x] & \text{if } S_F(x) > 0 \\ 0 & \text{otherwise,} \end{cases}$$

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where $S_F(x) = 1 - F(x)$ is the survival function. Note that $e(x)$ is the usual mean if $x = 0$. $e(x)$ uniquely determines the life distribution F via an inversion formula. Thus estimating $e(x)$ may lead to both determining classes of life time distributions and testing these classes. The MRLF plays very important role in the area of engineering, medical sciences, and many other fields. Hence there have been many works on estimating $e(x)$ (for example, Ghorai and Rejtö(1987), Choobineh and Park(1990)). If a set of n independent random survival times $X_1, \cdots, X_n$ from n units is given, then an intuitive empirical estimator, denoted by $\widehat{e}^{EM}(x)$, can be defined as

$$\widehat{e}^{EM}(x) = \frac{1}{\widehat{S}_n(x)} \int_x^\infty \widehat{S}_n(u)\, du,$$

where $\widehat{S}_n(t) = \frac{1}{n}\sum_{i=1}^n I[X_i > t]$ is the ratio of units of surviving beyond time t. Note that $I[\cdot]$ is an indicator function. In some situations, especially in medical studies where the period of study may be fixed or random, one can only observe that the right censored lifetimes $(Z_1, \delta_1), \cdots, (Z_n, \delta_n)$ instead of obtaining $X_1, \cdots, X_n$. Here $\delta_i = I[X_i \le Y_i]$ and $Z_i = X_i \wedge Y_i$ with Y_i representing the $i - th$ censoring random variable for $i = 1, \cdots, n$, respectively. It is also assumed that $Y_1, \cdots, Y_n$ are i.i.d. with continuous $G(y) = P[Y \le y]$ and are also independent of $X_1, \cdots, X_n$. Under the random censoring model, Kaplan-Meier(1958) estimator $\widehat{S}_F^{KM}$ and Nelson-Aalen(1969,1976) estimator, $\widehat{S}_F^{NA}$, for S_F were proposed and studied their properties. Let H be the distribution function of the Z_i's , given by $H = 1 - S_F S_G$ and $S_G = 1 - G$, and define (possibly infinite) T_F, T_G, T_H by

$$T_F = \inf\{t\colon F(t) = 1\},$$

etc.. Then $e(x)$ can be expressed in terms of $S_F(x)$ as

$$e(x) = \int_x^T \frac{S_F(u)}{S_F(x)}\, d(u) \qquad \text{for} \quad S_F(x) > 0, \tag{1.1}$$

where $T = \min(\ T_F, T_G\ )$.

Yang(1977) and Kumazawa(1987) proposed estimators $\widehat{e}^{NA}$ and $\widehat{e}^{KM}$ for e(x) based on $\widehat{S}_F^{NA}$ and $\widehat{S}_F^{KM}$,respectively. However, it is known that these estimators have some defiencies for the cases of small sample sizes or heavy censorship.

In this paper we propose an estimator $\widehat{e}^{SV}(x)$ of $e(x)$ in (1.1) based on (Z_1, δ_1), $\cdots$, (Z_n, δ_n), which is a functional form of the estimator $\widehat{e}_F^{SV}(x)$ that Susarla and Van Ryzin (1980) proposed. We also investigate asymptotic properties including strong consistency and the weak convergence on the whole interval on $[0, T_H)$. Some Monte Carlo simulation studies are carried out to check and to compare the performances of $\widehat{e}^{SV}(x)$, $\widehat{e}^{KM}(x)$ and $\widehat{e}^{NA}(x)$.

2. THE PROPOSED ESTIMATOR $\widehat{e}^{SV}$

Based on observations $(Z_1, \delta_1), \cdots, (Z_n, \delta_n)$, we first review the large sample properties of Susarla -Van Ryzin estimator, $\widehat{S}_F^{SV}(x)$ of $S_F(x)$, for $0 \leq x \leq Z_{(n)}$,

$$\widehat{S}_F^{SV}(x) = \frac{\sum_{j=1}^{n} I[Z_{(j)} > x]}{n} \prod_{\{j:Z_{(j)}\}} \left\{ \frac{n-j+2}{n-j+1} \right\}^{I[\delta_{(j)}=0, Z_{(j)} \leq x]} \tag{2.1}$$

which is obtained by producting the empirical survival function and reciprocal form of the estimator for S_G, where $Z_{(1)} \leq \cdots \leq Z_{(n)}$ are ordered Z_i's. Susarla and Van Ryzin(1978) showed that $\widehat{S}_F^{SV}(x)$ of $S_F(x)$ is mean square consistent and almost sure consistent, and also converges weakly to a mean zero Gaussian process.

Now based on $\widehat{S}_F^{SV}$, we propose an estimator $\widehat{e}^{SV}(x)$ as

$$\widehat{e}^{SV}(x) = \frac{1}{\widehat{S}_F^{SV}(x)} \int_x^{T_n} \widehat{S}_F^{SV}(u)\, du, \tag{2.2}$$

where $T_n = Z_{(n)}$.

First, the strong consistency of $\widehat{e}^{SV}(x)$ can be shown in the following theorem.

Theorem 1. If F and G are continuous, $1 - H(T) = S_F(T)S_G(T) > 0$, and $0 < T < T_H$, then

$$\sup_{x \leq T} |\widehat{e}^{SV}(x) - e(x)| \to 0 \text{ with probability 1.} \tag{2.3}$$

Proof. Since $\widehat{S}_F^{SV}$ converges to S_F uniformly in x $(\in [0, T_F))$ with probability 1, it can be shown easily. Thus the proof is omitted.

If the same conditions on the boundedness as in Susarla and Ryzin (1980) are imposed, one can also get the convergence rate in Theorem 1 as follows.

Corollary 1. If the a_n are positive constants such that

$$\sum_{n=1}^{\infty} \frac{a_n^2}{n^2 H^4(M)} < \infty, \tag{C1}$$

for $0 < 2s < 1$ and $0 < 2t < 1 < p$,

$$\sum_{n=1}^{\infty} \frac{a_n^p}{H^{2p}(M) n^{tp}} < \infty, \tag{C2}$$

and

$$\liminf\ n^s H(M) > 0, \tag{C3}$$

where $M \leq T_H$, then the convergence rate of

$$\hat{e}^{SV}_{M(n)}(x) = \frac{1}{\hat{S}^{SV}_F(x)} \int_x^M \hat{S}^{SV}_F(u)\, du$$

is, for any $x \in [0, M]$

$$o(max\{a_n^{-1}, log\ n/n^{\frac{1-2s}{2}}\} n^{\frac{1}{3}} S_G^{-1}(M)M).$$

Susarla and Van Ryzin(1978,1980) shows that the empirical process of survival function estimaor in (2.1) converges weakly to a Gaussian process $Z^*(x)$ as $n \to \infty$ with mean zero and covariance function

$$\sigma(x,y) \quad = \quad S_F(x)S_F(y) \int_0^{x \wedge y} S_F^{-2}(u) S_G^{-1} dF(u).$$

Under some regularity conditions of weak convergence of the process over the whole line given by Gill(1983), one can obtain the following theorem.

Theorem 2. Suppose that the distributions F and G satisfy the condition $\int_0^{T_H} h^2(u) \frac{1}{S_F^2(u) S_G(u)} dF(u) < \infty$, with $h(x) = \int_x^{T_H} S_F(u)du$. Then the stochastic process

$$\sqrt{n}[\hat{e}^{SV}(x) - e(x)],$$

for $0 \leq x \leq T_n$ converges weakly in $D[0, T_H)$ as $n \to \infty$ to a Gaussian process $B(x)$ with mean zero and covariance function

$$\frac{1}{S_F^2(x)} \int_x^{T_H} h^2(u) \frac{1}{S_F^2(u) S_G(u)} dF(u).$$

Proof. From the results of Susarla-Van Ryzin(1980) and Yang(1977), using the continuity theorem in Billingsley (Theorem 5.1,1968) one can obtain that the process of Theorem 2 converges weakly to

$$[S_F(x)]^{-2}\{S_F(x) \int_x^{T_H} Z^*(u)du - Z^*(x) \int_x^{T_H} S_F(u)du\}. \tag{2.4}$$

Since $\int_x^{T_H} Z^*(u)du = \int_x^{T_H} S_F(u) \frac{Z^*(u)}{S_F(u)} du$, letting $h(x) = \int_x^{T_H} S_F(u)du$ and using the integration by parts,

$$\begin{aligned}\int_x^{T_H} Z^*(u)du &= -\int_x^{T_H} \frac{Z^*(u)}{S_F(u)} dh(u) \\ &= \frac{Z^*(x)}{S_F(x)} \int_x^{T_H} S_F(u)du + \int_x^{T_H} h(u)d\left[\frac{Z^*(u)}{S_F(u)}\right].\end{aligned}$$

The above function h satisfies the conditions i) and ii) of the Theorem 2.1 of Gill(1983). Hence this completes the proof.

The limiting process $B(x)$ can be expressed as

$$\frac{1}{S_F(x)} \int_x^{T_H} h(u)d\left[\frac{Z^*(u)}{S_F(u)}\right].$$

Remark 1. If there is no censoring, then Theorem 2 is coincided with Theorem 1 in Yang (1977).

Remark 2. In Theorem 2, the asymptotic variance and the confidence band are obtained using the techniques of Gill(1983).

3. COMPARISONS OF ESTIMATORS $\hat{e}^{SV}, \hat{e}^{KM}$ AND $\hat{e}^{NA}$

It is common to select a few items for testing because the total test cost is generally a nondecreasing function of sample size, and practioners are inclined to minimize cost. Thus to compare the performances of estimators $\hat{e}^{SV}, \hat{e}^{KM}$ and $\hat{e}^{NA}$, some Monte Carlo simulations were carried out.

The simulation scheme is designed with various combinations of censoring patterns(10%, 20%, and 30%), different sample sizes(n = 10, 20, and 30), two survival distributions, exponential and Weibull, and two censoring distributions, uniform and exponential. Only the results for the cases of censoring patterns 10%, 20%, 30% and n = 10, 20, 30 are tabulated in Table 1, Table 2 and Table 3. The rest are available based upon request.

Note that the censoring ratio used for the simulation is the probabilistic censoring ratio and is not the exact one. The x's given as conditionals were obtained by the inverses of S_F, *i.e.*, $x = S_F^{-1}(0.9)$, $S_F^{-1}(0.8)$, $\cdots$, $S_F^{-1}(0.1)$. The number of replication for all cases were 1000 times and the mean, bias and mean square error(MSE's) were computed. For all cases the standard errors of MSE's are (approximately) less enough than 0.009.

From the results one can see the followings

(1) As Tables 1, 2 and 3 indicate, the MSE's of the proposed estimator $\widehat{e}^{SV}$ are consistently smaller than those of the $\widehat{e}^{KM}$ and $\widehat{e}^{NA}$ for small samples and heavy censoring cases.

(2) Also the estimator $\widehat{e}^{SV}$ performs always well far to the right tail of the life distribution in terms of bias.

(3) In Tables 1 and 2, the change of censoring distribution from uniform to exponential causes no essential changes in the results,*i.e.*, at small value of x, $\widehat{e}^{KM}$ is better, at the near middle points of x, $\widehat{e}^{NA}$ is better, and at the right tail parts of x, $\widehat{e}^{SV}$ is better in terms of bias. But in Table 2, $\widehat{e}^{NA}$ is overall better because it has always positive bias.

(4) Particularly, Table 3, for the case of Weibull life distribution shows that the estimator $\widehat{e}^{NA}$ is much better in terms of both bias and MSE.

Furthermore, unlike $\widehat{e}^{NA}$ and $\widehat{e}^{KM}$, the proposed estimator $\widehat{e}^{SV}$ can also be used to estimate the mean residual life time at the censoring point x. To illustrate this we generate Figure 1 by using the data in Kalbfleisch and Prentice(1980) for the squamous carcinoema in the mouth and throat. In Figure 1, the letters c's and u's denote the censoring and uncensoring points, respectively .

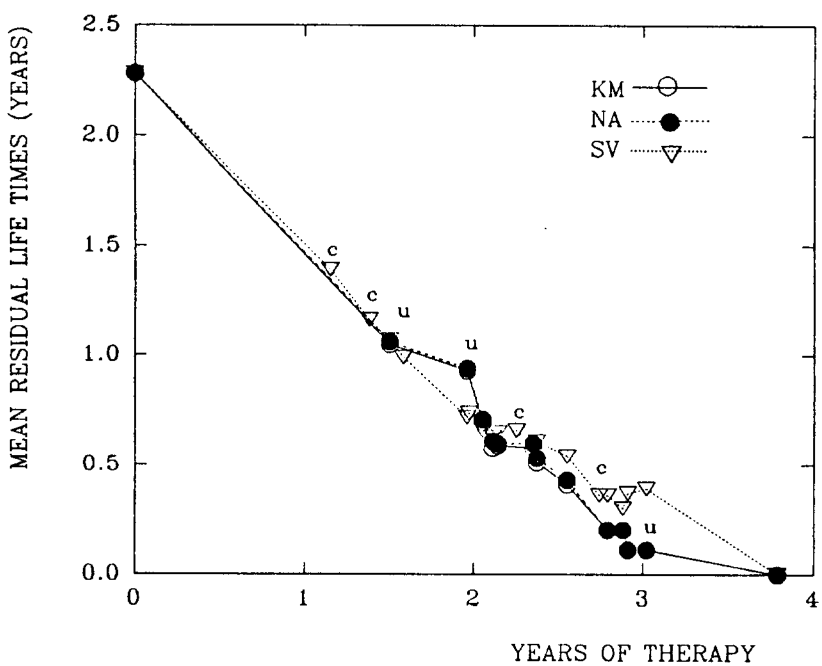

Figure 1. Mean Residual Life Times for Carcimona Patients

Table 1. The values of $\widehat{e}^{KM}(x)$, $\widehat{e}^{SV}(x)$, and $\widehat{e}^{NA}(x)$ for various n, x and censoring ratios when $S_F(x) = \text{Exp}(1), S_G(x) = \text{Uniform}(\lambda)$

			true values of $e(x)$ at age x(conditional points)								
n	r%		1(.105)	1(.223)	1(.357)	1(.511)	1(.693)	1(.916)	1(1.204)	1(1.610)	1(2.304)
10	10%	KM	.963(.115)	.962(.138)	.959(.167)	.953(.200)	.931(.223)	.913(.283)	.900(.348)	.869(.397)	.890(.558)
		NA	1.058(.152)	1.054(.171)	1.055(.208)	1.052(.250)	1.031(.275)	1.011(.348)	.990(.413)	.947(.458)	.941(.596)
		SV	.943(.103)	.942(.121)	.943(.147)	.941(.174)	.926(.198)	.924(.270)	.920(.340)	.907(.393)	.935(.543)
	20%	KM	.913(.146)	.897(.164)	.886(.182)	.860(.224)	.851(.271)	.845(.307)	.842(.328)	.781(.381)	.823(.497)
		NA	.973(.173)	.971(.184)	.963(.201)	.938(.252)	.930(.307)	.926(.351)	.925(.383)	.854(.428)	.894(.580)
		SV	.888(.122)	.874(.138)	.866(.154)	.850(.193)	.839(.230)	.841(.283)	.855(.302)	.829(.343)	.854(.447)
	30%	KM	.867(.160)	.853(.192)	.842(.226)	.804(.263)	.760(.276)	.752(.288)	.719(.322)	.713(.342)	.712(.274)
		NA	.935(.164)	.914(.199)	.905(.244)	.866(.281)	.823(.291)	.817(.306)	.786(.340)	.781(.363)	.761(.279)
		SV	.835(.120)	.828(.140)	.821(.165)	.796(.198)	.769(.243)	.773(.261)	.747(.291)	.736(.329)	.715(.278)
20	10%	KM	.992(.068)	.998(.077)	.986(.091)	.989(.115)	.984(.131)	.977(.171)	.978(.247)	.966(.314)	.949(.501)
		NA	1.061(.083)	1.058(.093)	1.063(.112)	1.072(.140)	1.074(.165)	1.072(.213)	1.078(.308)	1.066(.386)	1.040(.572)
		SV	.976(.062)	.947(.069)	.970(.082)	.972(.101)	.970(.119)	.965(.159)	.968(.226)	.967(.287)	.952(.481)
	20%	KM	.944(.064)	.941(.077)	.936(.094)	.931(.116)	.916(.147)	.899(.192)	.878(.257)	.839(.303)	.755(.328)
		NA	1.001(.071)	1.005(.087)	1.004(.106)	1.002(.132)	.991(.167)	.977(.218)	.957(.288)	.918(.342)	.834(.364)
		SV	.914(.057)	.910(.067)	.905(.079)	.900(.096)	.887(.120)	.875(.154)	.854(.206)	.849(.296)	.807(.369)
	30%	KM	.906(.083)	.892(.098)	.877(.115)	.863(.145)	.836(.181)	.806(.221)	.759(.270)	.718(.272)	.611(.333)
		NA	.957(.089)	.946(.103)	.933(.121)	.921(.152)	.896(.189)	.867(.230)	.821(.281)	.782(.278)	.666(.322)
		SV	.871(.070)	.857(.081)	.844(.094)	.833(.113)	.812(.139)	.799(.180)	.769(240)	.748(.298)	.717(.330)
30	10%	KM	.993(.043)	.994(.050)	.996(.058)	.995(.070)	.996(.089)	.997(.117)	.993(.170)	.942(.219)	.930(.335)
		NA	1.044(.051)	1.051(.059)	1.057(.069)	1.062(.084)	1.071(.109)	1.080(.145)	1.085(.210)	1.039(.266)	1.025(.395)
		SV	.977(.040)	.978(.046)	.978(.052)	.976(.062)	.977(.079)	.977(.102)	.972(.145)	.927(.203)	.930(.329)
	20%	KM	.975(.048)	.976(.055)	.971(.065)	.962(.080)	.949(.098)	.926(.124)	.919(.183)	.871(.255)	.832(.340)
		NA	1.023(.054)	1.023(.064)	1.028(.075)	1.024(.092)	1.015(.112)	.998(.140)	.996(.209)	.951(.282)	.917(.391)
		SV	.949(.041)	.948(.048)	.943(.056)	.934(.068)	.922(.083)	.903(.105)	.904(.156)	.862(.215)	.829(.324)
	30%	KM	.926(.054)	.915(.061)	.905(.076)	.887(.095)	.867(.122)	.835(.159)	.788(.203)	.726(.255)	.671(.275)
		NA	.969(.060)	.961(.063)	.953(.078)	.939(.098)	.923(.126)	.893(.162)	.848(.206)	.789(.256)	.738(.273)
		SV	.890(.048)	.879(.054)	.868(.064)	.853(.079)	.836(.099)	.809(.127)	.773(.174)	.726(.239)	.691(.260)

* KM, NA and SV represent the values of $\widehat{e}^{KM}(x)$, $\widehat{e}^{NA}(x)$ and $\widehat{e}^{SV}(x)$, respectively.
* Values in the parenthesis are MSE's.
* "True values" denote values of $e(x)$ at age x.
* n denotes a sample size.
* r denotes a censoring ratio.

Table 2. The values of $\widehat{e}^{KM}(x)$, $\widehat{e}^{SV}(x)$, and $\widehat{e}^{NA}(x)$ for various n, x and censoring ratios when $S_F(x) = \text{Exp}(1), S_G(x) = \text{Exp}(\lambda)$

			true values of $e(x)$ at age x(conditional points)								
n	r%		1(.105)	1(.223)	1(.357)	1(.511)	1(.693)	1(.916)	1(1.204)	1(1.610)	1(2.304)
	10%	KM	.963(.132)	.961(.150)	.962(.197)	.942(.237)	.925(.277)	.891(.332)	.894(.373)	.899(.414)	.890(.551)
		NA	1.066(.173)	1.058(.190)	1.054(.242)	1.055(.289)	1.038(.335)	1.018(.396)	.976(.437)	.971(.480)	.950(.601)
		SV	.945(.112)	.943(.125)	.944(.159)	.945(.192)	.930(.226)	.924(.285)	.919(.340)	.939(.463)	.955(.558)
	20%	KM	.879(.114)	.861(.139)	.840(.163)	.805(.194)	.769(.222)	.726(.255)	.689(.271)	.649(.311)	.588(.307)
10		NA	.928(.126)	.924(.146)	.903(.167)	.868(.197)	.830(.222)	.784(.255)	.742(.266)	.698(.301)	.619(.294)
		SV	.879(.100)	.865(.121)	.854(.139)	.835(.166)	.818(.195)	.792(.228)	.768(.245)	.725(.281)	.639(.288)
	30%	KM	.785(.130)	.759(.150)	.722(.174)	.679(.208)	.630(.248)	.590(.267)	.510(.329)	.396(.438)	.316(.494)
		NA	.826(.123)	.803(.141)	.764(.163)	.721(.194)	.671(.232)	.629(.249)	.545(.306)	.420(.416)	.329(.478)
		SV	.812(.101)	.797(.114)	.781(.132)	.750(.155)	.710(.187)	.685(.204)	.606(.252)	.516(.324)	.321(.489)
	10%	KM	.984(.062)	.981(.069)	.978(.081)	.977(.095)	.969(.118)	.958(.151)	.945(.213)	.930(.268)	.834(.316)
		NA	1.048(.074)	1.049(.083)	1.050(.096)	1.056(.114)	1.053(.141)	1.048(.181)	1.039(.254)	1.023(.318)	.916(.361)
		SV	.972(.057)	.968(.064)	.965(.076)	.966(.088)	.959(.111)	.949(.138)	.943(.193)	.941(.252)	.875(.311)
	20%	KM	.912(.058)	.906(.068)	.895(.081)	.877(.098)	.852(.123)	.816(.149)	.769(.189)	.683(.249)	.572(.292)
20		NA	.961(.058)	.957(.068)	.949(.081)	.934(.098)	.911(.123)	.876(.147)	.827(.184)	.737(.240)	.616(.270)
		SV	.904(.053)	.900(.062)	.891(.071)	.879(.085)	.862(.101)	.835(.127)	.807(.170)	.742(.229)	.669(.262)
	30%	KM	.843(.069)	.822(.085)	.795(.101)	.760(.128)	.713(.164)	.651(.214)	.565(.284)	.481(.338)	.248(.587)
		NA	.872(.065)	.856(.078)	.830(.0924)	.796(.118)	.748(.152)	.684(.199)	.596(.265)	.511(.314)	.266(.563)
		SV	.852(.061)	.836(.072)	.816(.084)	.792(.102)	.763(.124)	.722(.156)	.651(.206)	.589(.250)	.416(.384)
	10%	KM	.989(.044)	.990(.049)	.990(.059)	.988(.069)	.987(.087)	.983(.112)	.976(.162)	.933(.232)	.932(.323)
		NA	1.042(.051)	1.044(.059)	1.049(.070)	1.052(.083)	1.058(.105)	1.061(.136)	1.063(.198)	1.025(.271)	1.022(.379)
		SV	.977(.040)	.976(.046)	.976(.054)	.973(.062)	.973(.078)	.970(.099)	.967(.146)	.926(.204)	.922(.307)
	20%	KM	.942(.042)	.936(.048)	.926(.058)	.911(.071)	.888(.086)	.852(.116)	.816(.149)	.723(.207)	.617(.252)
30		NA	.981(.042)	.979(.049)	.972(.059)	.960(.071)	.940(.085)	.908(.113)	.874(.147)	.779(.198)	.670(.233)
		SV	.929(.039)	.924(.045)	.915(.052)	.901(.062)	.882(.074)	.854(.097)	.828(.129)	.761(.179)	.678(.249)
	30%	KM	.862(.049)	.842(.058)	.821(.072)	.790(.092)	.749(.118)	.686(.167)	.596(.239)	.488(.323)	.285(.531)
		NA	.889(.044)	.871(.052)	.851(.065)	.821(.083)	.781(.108)	.717(.154)	.625(.222)	.516(.301)	.307(.502)
		SV	.864(.044)	.848(.051)	.831(.061)	.806(.075)	.777(.092)	.738(.123)	.671(.175)	.588(.243)	.411(.387)

* KM, NA and SV represent the values of $\widehat{e}^{KM}(x)$, $\widehat{e}^{NA}(x)$ and $\widehat{e}^{SV}(x)$, respectively.
* Values in the parenthesis are MSE's.
* "True values" denote values of $e(x)$ at age x.
* n denotes a sample size.
* r denotes a censoring ratio.

Table 3. The values of $\widehat{e}^{KM}(x)$, $\widehat{e}^{SV}(x)$, and $\widehat{e}^{NA}(x)$ for various n, x and censoring ratios when $S_F(x) = \text{Weib}(1.15, 2), S_G(x) = \text{Exp}(\lambda)$

			true values of $e(x)$ at age x(conditional points)								
n	r%		.551(.281)	.483(.409)	.437(.517)	.399(.619)	.367(.721)	.337(.829)	.309(.950)	.280(1.098)	.244(1.314)
	10%	KM	.543(.016)	.477(.017)	.431(.019)	.396(.020)	.359(.021)	.328(.023)	.298(.025)	.267(.026)	.243(.025)
		NA	.555(.019)	.514(.020)	.466(.022)	.431(.025)	.394(.025)	.360(.028)	.327(.030)	.291(.030)	.258(.028)
		SV	.536(.015)	.471(.016)	.427(.017)	.392(.019)	.357(.019)	.328(.023)	.300(.025)	.269(.026)	.243(.025)
	20%	KM	.539(.021)	.467(.022)	.420(.022)	.378(.024)	.345(.026)	.324(.027)	.299(.025)	.266(.027)	.249(.025)
10		NA	.557(.024)	.501(.024)	.453(.025)	.410(.028)	.377(.030)	.355(.032)	.329(.030)	.290(.031)	.269(.029)
		SV	.525(.019)	.456(.020)	.409(.020)	.371(.022)	.338(.024)	.316(.026)	.297(.024)	.267(.025)	.247(.024)
	30%	KM	.535(.023)	.468(.025)	.424(.026)	.377(.027)	.337(.028)	.314(.025)	.283(.024)	.251(.021)	.211(.019)
		NA	.559(.026)	.499(.027)	.455(.029)	.407(.031)	.367(.031)	.343(.029)	.309(.028)	.276(.024)	.230(.022)
		SV	.513(.019)	.451(.020)	.409(.022)	.368(.023)	.332(.025)	.313(.023)	.283(.023)	.251(.022)	.214(.021)
	10%	KM	.552(.009)	.483(.009)	.436(.010)	.401(.010)	.369(.011)	.339(.014)	.315(.018)	.286(.019)	.245(.023)
		NA	.572(.010)	.508(.010)	.462(.011)	.428(.013)	.397(.014)	.369(.017)	.345(.022)	.315(.024)	.269(.028)
		SV	.547(.008)	.479(.008)	.432(.009)	.397(.010)	.366(.011)	.337(.013)	.311(.017)	.284(.018)	.244(.022)
	20%	KM	.542(.009)	.475(.009)	.429(.010)	.393(.011)	.359(.012)	.329(.014)	.298(.016)	.268(.019)	.232(.021)
20		NA	.565(.009)	.500(.010)	.455(.011)	.420(.013)	.387(.014)	.358(.017)	.327(.019)	.295(.023)	.256(.025)
		SV	.531(.008)	.465(.008)	.420(.009)	.384(.010)	.351(.011)	.322(.013)	.293(.015)	.265(.019)	.229(.020)
	30%	KM	.543(.011)	.473(.012)	.425(.013)	.389(.015)	.354(.017)	.324(.018)	.292(.022)	.261(.024)	.231(.022)
		NA	.566(.012)	.499(.013)	.451(.014)	.416(.017)	.381(.019)	.351(.021)	.319(.025)	.286(.028)	.254(.025)
		SV	.524(.010)	.457(.010)	.409(.011)	.375(.012)	.342(.014)	.316(.016)	.287(.020)	.258(.023)	.230(.020)
	10%	KM	.550(.005)	.483(.005)	.438(.005)	.401(.006)	.370(.007)	.342(.009)	.315(.011)	.274(.012)	.242(.017)
		NA	.567(.006)	.502(.006)	.457(.007)	.422(.007)	.393(.009)	.367(.010)	.341(.014)	.301(.015)	.267(.020)
		SV	.546(.006)	.480(.006)	.434(.006)	.398(.006)	.367(.007)	.339(.008)	.312(.010)	.273(.012)	.241(.017)
	20%	KM	.550(.006)	.485(.007)	.438(.007)	.400(.008)	.365(.008)	.333(.009)	.270(.012)	.270(.015)	.832(.018)
30		NA	.569(.007)	.505(.008)	.459(.008)	.422(.009)	.389(.010)	.358(.011)	.335(.015)	.297(.018)	.253(.022)
		SV	.542(.006)	.477(.006)	.431(.007)	.393(.007)	.359(.008)	.327(.009)	.304(.012)	.267(.014)	.229(.017)
	30%	KM	.545(.007)	.477(.006)	.429(.007)	.390(.008)	.357(.010)	.325(.012)	.289(.013)	.257(.016)	.213(.015)
		NA	.565(.008)	.497(.008)	.451(.008)	.413(.010)	.381(.011)	.350(.013)	.315(.015)	.281(.018)	.235(.016)
		SV	.532(.007)	.464(.007)	.418(.007)	.380(.008)	.347(.009)	.316(.010)	.286(.012)	.256(.015)	.213(.014)

* KM, NA and SV represent the values of $\widehat{e}^{KM}(x)$, $\widehat{e}^{NA}(x)$ and $\widehat{e}^{SV}(x)$, respectively.
* Values in the parenthesis are MSE's.
* "True values" denote values of $e(x)$ at age x.
* n denotes a sample size.
* r denotes a censoring ratio.

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REFERENCES

(1) Aalen, O. (1976). Nonparametric inference inconnect with multiple decrement models. *Scandinavian Journal of Statistics,* 3, 15-27.

(2) Billingsley, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

(3) Breslow, N. E. and Crowley, J.(1974). A large sample study of the life table and product limit estimators under random censorship. *The Annals of Statistics*, 2, 437-453.

(4) Choobineh, F. and Park, D. H.(1990). A nonparametric small sample estimator of mean residual life. *Journal of The Korean Statistical Society*, 16, 1,80-87.

(5) Ghorai, J. K. and Rejtö, L. (1987). Estimation of mean residual life with censored data under the proportional hazard model. *Communications in Statistics - Theory and Methods*, 16, 2097-2114.

(6) Gill, R.D.(1983). Large sample behavior of the product-limit estimator on the whole line. *The Annals of Statistics*, 11, 49-58.

(7) Kabfleisch, T.D. and Prentice, R.L.(1980). *The Statistical Analysis of Failure Time Data.* Wiley, New York.

(8) Kaplan, E.L. and Meier, P.(1958). Nonparametric estimation from incomplete observations. *Journal of the American Statistical Association,* 53, 457-481.

(9) Kumazawa, Y.(1987). A note on estimator of life expectancy with random censorship. *Biometrika*, 74, 3, 655-658.

(10) Nelson, W. (1969). Hazard plotting for incomplete failure data. *Journal of Quality Technology,* 1, 27-52.

(11) Nelson, W.(1972). Theory and application of hazard plotting for censored failure data. *Technometrics*, 14, 945-966.

(12) Susarla, V. and Van Ryzin, J.(1978). Large sample theory for a Bayesian nonparametric survival curve estimator based on censored samples. *The Annals of Statistics*, 6, 755-768.

(13) Susarla, V. and Van Ryzin, J.(1980). Large sample theory for an estimator of the mean survival time from censored samples. *The Annals of Statistics*, 8, 1002-1016.

(14) Yang, G.(1977). Life expectancy under random censorship. *Stochastic Processes and Their Applications* , 6, 33-39.