Journal of the Korean
Statistical Society
Vol. 22, No. 1, 1993

A Note on Estimation under Discrete Time Observations in the Simple Stochastic Epidemic Model †

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ABSTRACT

We consider two estimators of the infection rate in the simple stochastic epidemic model. It is shown that the maximum likelihood estimator of the infection rate under the discrete time observation does not have the moment of any positive order. Some properties of the Choi-Severo estimator, an approximation to the maximum likelihood estimator, are also investigated.

KEYWORDS: Simple stochastic epidemic model, Infection rate, Discrete observations, Approximation to the m.l.e.

1. INTRODUCTION

In the simple stochastic epidemic model, we assume that there exists a homogeneously mixing group of N individuals consisting of a infectives and

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† This research was supported by a Korea Research Foundation Grant.

$N-a$ susceptibles. Let $X(t)$ and $Y(t)$ represent the numbers of susceptibles and infectives, respectively, at time $t \geq 0$. For this model the infinitesimal transition probabilities are given by:

$$P\{X(t+h)=i|X(t)=x\} = \begin{cases} \beta x(N-x)h+o(h) & \text{if } i=x-1 \\ 1-\beta x(N-x)h+o(h) & \text{if } i=x \\ o(h) & \text{if } i<x-1 \end{cases}$$

and $P\{X(t+h)=0|X(t)=0\}=1$. Here β is the infection rate and $x = N-a,\ N-a-1,\ \cdots,\ 1$. We define $P_x(t)=P\{X(t)=x|X(0)=N-a\}$. Then the differential-difference equations associated with this model can be obtained (see Bailey (1975)).

Bailey (1975) solved this system of differential-difference equations by a Laplace transformation technique but the result was very complicated. Severo (1967) developed an iterative solution for the state probabilities. Yang (1972) and Billard, Lacayo and Langberg (1979) used interinfection times to get the desired probabilities.

Since the obtained probabilities depend on the value of the infection rate β, it is natural to carry out parameter estimation. For estimation in the simple stochastic epidemic model, sampling at predetermined time points $t_0<t_1<\cdots<t_n$ is a practical sampling scheme. Under this sampling scheme, an explicit form of the maximum likelihood estimator of β is not known. Here we denote the data set as

$$\{(t_0,y_0)=(0,a),(t_1,y_1),\cdots,(t_n,y_n)\} \tag{1.1}$$

where y_i is the number of infectives at time t_i and $t_0<t_1<\cdots<t_n$ together with $y_0 \leq y_1 \leq \cdots \leq y_n$. Because of the Markov property, the likelihood function of β is then given by

$$L(\beta)=P\{Y(t_1)=y_1|Y(t_0)=y_0\}\cdots P\{Y(t_n)=y_n|Y(t_{n-1})=y_{n-1}\} \tag{1.2}$$

Let the maximum likelihood estimator of β with this discrete type data be denoted by $\hat{\beta}_{MD}$. With the restriction $y_n<N$, Hill and Severo (1969) and Kryscio (1972) suggested approximations to $\hat{\beta}_{MD}$. Without restriction of Hill and Severo (1969) Choi and Severo (1988) suggested an approximation

$$\hat{\beta}_{CS} = 2(y_n - a) / \sum_{i=1}^{n} (t_i - t_{i-1})\{y_i(N - y_i) + y_{i-1}(N - y_{i-1})\},$$

which has some advantages over the previous approximations.

However very little is known on $\hat{\beta}_{MD}$ in the literature. In the next section we investigate some properties of $\hat{\beta}_{MD}$ and $\hat{\beta}_{CS}$ under the sampling scheme.

2. ESTIMATION OF β UNDER THE FIXED SAMPLING SCHEME

For the observation of an epidemic process at a discrete set of time points, we do not have an explicit form for $\hat{\beta}_{MD}$. Therefore the expectation and the variance of the maximum likelihood estimator of β can not be obtained directly. However we prove that the expectation of $\hat{\beta}_{MD}$ does not exist by showing that the probability of all susceptibles becoming infectives by time t_1 is positive and the value of $\hat{\beta}_{MD}$ in that case is positive infinity. Indeed we show that $\hat{\beta}_{MD}$ possesses no moments of any positive order.

Lemma 1. For data $\{(0, a), (t_1, N), \cdots, (t_n, N)\}$, $\hat{\beta}_{MD} = \infty$.

Proof. Let $Z_1, \cdots, Z_{N-a}$ be the interinfection times. Then $Z_1, \cdots, Z_{N-a}$ are independent, exponentially distributed random variables with rate parameters $\beta_{g_i} = \beta(a + i - 1)(N - a - i + 1)$, respectively. The likelihood function of β then becomes $L(\beta) = P\{Z_1 + \cdots + Z_{N-a} < t_1\}$. We consider two cases.

Case I: $(N/2) \leq a < N$. In this case it is clear that $g_1 > \cdots > g_{N-a}$. From Theorem 1 of Billard, Lacayo and Langberg (1979), it is easily shown that $\hat{\beta}_{MD} = \infty$ is a maximum likelihood estimate of β. Clearly, this is the unique such estimate.

Case II: $1 \leq a < N/2$. Let $m = [N/2]$ and $r = N - a - [N/2]$ where $[x]$ is the greatest integer not exceeding x. Let $\mu_i = g_{r+i}, i = 1, \cdots, m$ and $\tilde{\mu}_i = g_{r-i+1}, i = 1, \cdots, r$. Then $\mu_i \neq \mu_j$ for $i \neq j, i = 1, \cdots, m$ and $j = 1, \cdots, m$ and $\tilde{\mu}_i \neq \tilde{\mu}_j$ for $i \neq j, i = 1, \cdots, r$ and $j = 1, \cdots, r$. Now we rearrange $\mu_i, i = 1, \cdots, m$ so that $\tilde{\mu}_i = \mu_i$ for $i = 1, \cdots, r$. Then by use of Theorem 2 of Billard, Lacayo and Langberg (1979), we can easily show that $\hat{\beta}_{MD} = \infty$ is the unique maximum likelihood estimate of β.

Theorem 2. For any positive real number λ, the moment of $\hat{\beta}_{MD}$ of order λ does not exist.

Proof. It is clear that $P\{Y(t_0) = a, Y(t_1) = N, \cdots, Y(t_n) = N\} > 0$ for all $t_1 > 0$. Let $b(s_0)$ be the value of the maximum likelihood estimator of β based on the data $s_0 = \{(t_0, y_0), (t_1, y_1), \cdots, (t_n, y_n)\}$. Then,

$$E(\hat{\beta}_{MD}^{\lambda}) \geq \{(b(s_0)\}^{\lambda} P\{Y(0) = a, Y(t_1) = N, \cdots, Y(t_n) = N\}.$$

Now the right-hand-side of the inequality is infinite, and thus the proof is finished.

Due to the intractability of calculations, it is not attractive to try to get the expectation of $\hat{\beta}_{CS}$. But in contrast to $\hat{\beta}_{MD}$, the expectation of $\hat{\beta}_{CS}$ exists and can be bounded as in the following theorem.

Theorem 3. Let p and q be defined by $p = P\{Y(t_n) = a\} = P(Z_1 > t_n)$ and $q = P\{Y(t_1) = a\} = P(Z_1 > t_1)$. If $0 < a \leq N/2$, then

$$(\frac{2}{N})^2 a(N-a)\beta\frac{p-1}{\log p} \leq E(\hat{\beta}_{CS}) \leq 2(N-a)\beta\frac{p-1}{\log q}. \tag{2.1}$$

If $N/2 < a < N$, then

$$\beta\frac{p-1}{\log p} \leq E(\hat{\beta}_{CS}) \leq 2(N-a)\beta\frac{p-1}{\log q}. \tag{2.2}$$

Proof. Since $P(Z_1 > t) = \exp(-a(N-a)\beta t)$ for $t > 0$, $p = \exp(-a(N-a)\beta t_n)$, and $q = \exp(-a(N-a)\beta t_1)$, we have $t_n = -(\log p)/(a(N-a)\beta)$ and $t_1 = -(\log q)/(a(N-a)\beta)$. On the other hand, it is clear that $E(\hat{\beta}_{CS}) \leq 2(1-p)/(t_1 a)$. By substituting $t_1 = -(\log q)/(a(N-a)\beta)$, we get the upper bound of (2.1). For the lower bound we consider two cases. The first case is for $a \leq N/2$. Since $t_n = -(\log p)/(a(N-a)\beta)$, we have the lower bound in (2.1). For the case $a > N/2$, the result is easily obtained.

The following corollary tells us lower bounds and upper bounds of $E(\hat{\beta}_{CS})$ when t_n converges to 0 or t_1 diverges to infinity.

Corollary 4.

(1) Let $t_n = Kt_1,\ K > 1$. Then, for $0 < a < N/2$,

$$b(\frac{2}{N})^2 a(N-a)\beta \le \lim_{t_n \to 0} E(\hat{\beta}_{CS}) \le 2K(N-a)\beta, \tag{2.3}$$

and, for $N/2 \le a < N$,

$$\beta \le \lim_{t_n \to 0} E(\hat{\beta}_{CS}) \le 2K(N-a)\beta. \tag{2.4}$$

(2) $\lim_{t_1 \to \infty} E(\hat{\beta}_{CS}) = 0$.

Proof. (1) That t_n approaches 0 implies $p \to 1$. Since $(p-1)/\log p$ is monotone increasing in p and converges to 1 as $p \to 1$ for $0 < p < 1$, we easily obtain the left hand inequalities in (2.3) and (2.4).
Since $p = \exp(-a(N-a)\beta K t_1)$ and $q = \exp(-a(N-a)\beta t_1)$, we have $(p-1)/\log q = (\exp(-a(N-a)\beta K t_1) - 1)/(-a(N-a)\beta t_1) \to K$ as $t_1 \to 0$. Thus we get the right hand inequalities in (2.1) and (2.2).
(2) That $0 < p \le q < 1$, $p \to 0$ and $q \to 0$ as $t_1 \to \infty$, we have the result.

REFERENCES

(1) Bailey, N. T. J. (1975). *The Mathematical Theory of Infectious Diseases and its Applications, second edition.* Hafner Press, New York.

(2) Billards, L., Lacayo, H. and Langberg, N. A. (1979). A new look at the simple epidemic process. *Journal of Applied Probability*, Vol. 16,198-202.

(3) Choi, Y. J. and Severo, N. C. (1988). An approximation for the maximum likelihood estimator of the infection rate in the simple stochastic epidemic. *Biometrika*, Vol. 75, 392-394.

(4) Hill, R. T. and Severo, N. C. (1969). The simple stochastic epidemic for small populations with one or more initial infectives. *Biometrika*, Vol. 56, 183-196.

(5) Kryscio, R. J. (1972). On estimating the infection rate of the simple stochastic epidemic. *Biometrika*, Vol. 59, 213-214.

(6) Kryscio, R. J. and Severo, N. C. (1975). Computational and estimation procedures in multidimensional right-shift processes and some applications. *Advances in Applied Probability*, Vol. 7, 349-382.

(7) Severo, N. C. (1967). Two theorems on solutions of difference equations and applications to epidemic theory. *Journal of Applied Probability*, Vol. 4, 271-280.

(8) Yang, G. (1972). On the probability of some stochastic epidemic models. *Theoretical Population Biology*, Vol. 3, 448-459.