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A Note on Tests for Seasonal Unit Roots in the Presence of Deterministic Trends†

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ABSTRCT

In this paper we show that the results of Ahn and Cho (1992) can be applied to a more general class of seasonal models, especially models with autocorrelated errors. Employing the idea of the "two-step estimation" method, we provide test statistics which are easy to compute and have the same asymptotic properties as those in Ahn and Cho (1992) for seasonal unit roots. A numerical example is presented to illustrate the methods and concepts. The power of the test statistics for finite samples is examined through a Monte Carlo sampling experiment.

KEYWORDS: Seasonal unit roots, Deterministic trends, Lagrange multiplier test, Brownian bridge

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1. INTRODUCTION

For time series data which exhibit nonstationarity, it is essential to identify if the nonstationarity is attributable to either a deterministic trend or a stochastic trend, or both. This is because the implication concerning the behavior and forecasts of the time series can be quite different, see Nelson and Plosser (1982). Since the work of Dickey and Fuller (1979, 1981) for nonseasonal time series and Dickey, Hasza, and Fuller (1984) for seasonal time series, identification of the type of the trends has been carried under the frame work of hypothesis testing for unit roots, both regular and seasonal. However, variants of the "Dickey-Fuller test" for unit roots are designed such that models under the null and alternative hypotheses are assumed to have no deterministic trends. Moreover, the test statistics of the variants of the "Dickey-Fuller test" have different asymptotic properties depending on whether the underlying process has a deterministic trend or not, see Nankervis and Savin (1984). Therefore, in order to employ these tests properly, *apriori* information about the existence of a deterministic trend is required. Often, such information may not be readily available.

Especially, for seasonal nonstationary time series empirical studies have indicated that deterministic trends tend to well describe seasonal nonstationarity compared with stochastic trends because of the consistent and repetitive effects of, for example, holidays and the weather, see Barskey and Miron (1989). Therefore, it may not be appropriate to assume no deterministic trends and to use one of the test statistics for a seasonal unit root in Dickey, Hasza, and Fuller (1984), when no *apriori* information about the nonexistence of deterministic trend is available.

Extending the results of Ahn (1992) by applying the Lagrange multiplier (LM) principle, Ahn and Cho (1992) developed test statistics for a seasonal unit root and obtained their asymptotic properties, which are invariant to the existence of deterministic trends, of the following model which can accommodate the possibility of both types of the seasonal trends.

$$Y_t = \sum_{j=1}^{s}(\alpha_j + \beta_j\tau)\delta_{jt} + N_t, \tag{1.1}$$

$$N_t = \rho N_{t-s} + \epsilon_t,$$

where the ϵ_t are independent and normally distributed random variables with $E(\epsilon_t) = 0$ and $Var(\epsilon_t) = \sigma_\epsilon^2$, the δ_{jt} are seasonal dummy variables such that $\delta_{jt} = 1$ if $j \equiv t \ (mod\ s)$, and 0 otherwise, and $\tau = [(t-1)/s + 1]$ with $[x]$ denoting the largest integer no larger than x.

However, in practice the error term ϵ_t are more likely to be autocorrelated than to be independent. In this paper in order to account for the autocorrelation of the ϵ_t we assume that the ϵ_t is an autoregressive process of order p, AR(p), and show that the results in Ahn and Cho (1992) can be applied.

2. MAIN RESULTS

We consider the following model

$$Y_t = \sum_{j=1}^{s} (\alpha_j + \beta_j \tau)\delta_{jt} + N_t, \tag{2.1}$$

$$\phi(B)(1 - \rho B^s)N_t = e_t,$$

where the e_t are independent and normally distributed random variables with $E(e_t) = 0$ and $Var(e_t) = \sigma^2$, the δ_{jt} and τ are as defined in (1.1), and $\phi(B) = 1 - \sum_{i=1}^{p} \phi_i B^i$ is such that the roots of $\phi(B) = 0$ are all greater than one in absolute value.

This model represents a case where the ϵ_t in (1.1) follow a stationary AR(p), that is, $\phi(B)\epsilon_t = e_t$, thus covers more general class of seasonal models. Under the null hypothesis $\rho = 1$, the seasonally differenced noise series $N_t - N_{t-s}$ follow an AR(p) and the Y_t is nonstationary with the stochastic trend. The seasonally differenced series $Z_t = Y_t - Y_{t-s}$ after being adjusted for the seasonal effects also follow the same AR(p), that is, $\phi(B)(Z_t - \sum_{j=1}^{s} \beta_j \delta_{jt}) = e_t$. Nonzero β_j represents a deterministic seasonal trend for the j-th season.

When the ϕ_i in $\phi(B)$ are known, we can transform Y_t and rewrite Model (2.1) as

$$y_t = \sum_{j=1}^{s} (a_j + b_j \tau)\delta_{jt} + n_t, \tag{2.2}$$

$$(1 - \rho B^s)n_t = e_t,$$

where $y_t = \phi(B)Y_t$, $n_t = \phi(B)N_t$, $\sum a_j \delta_{jt} = \phi(B) \sum \alpha_j \delta_{jt}$, and $\sum b_j \delta_{jt} = \phi(B) \sum \beta_j \delta_{jt}$. Since Model (2.2) takes the exactly same form as Model (1.1) with the independent error e_t, for the test of a seasonal unit root we can directly apply the results in Ahn and Cho (1992). In practice since the ϕ_i are unknown, we use consistent estimators for the ϕ_i. By fitting the autoregressive model to the seasonally adjusted Z_t, we can obtain consistent estimators of the ϕ_i under the null hypothesis H_0 : $\rho = 1$. This suggests the following testing procedure based on the "two-step" estimation method.

1. Fit the following model to the seasonally differenced series $Z_t = Y_t - Y_{t-s}$ and obtain consistent estimator $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_p)'$ of ϕ.

$$\phi(B)Z_t = \sum_{j=1}^{s} \omega_j \delta_{jt} + e_t$$

2. Obtain $\tilde{y}_t$, $\tilde{b}_j(\tilde{\phi})$, and $\tilde{a}_j^n(\tilde{\phi})$ by

$$\tilde{y}_t = \tilde{\phi}(B)Y_t,$$

$$\tilde{b}_j(\tilde{\phi}) = (\tilde{y}_{(T_j-1)s+j} - \tilde{y}_j)/(T_j - 1),$$

$$\tilde{a}_j^n(\tilde{\phi}) = \tilde{y}_j - \tilde{b}_j(\tilde{\phi}),$$

where $\tilde{\phi}(B) = 1 - \sum_{i=1}^{p} \tilde{\phi}_i B^i$ and T_j the length of $\tilde{y}_t$ corresponding to the j-th season.

3. Fit the following simple linear regression model

$$\tilde{w}_t = \theta \tilde{r}_{t-s} + c_t, \tag{2.3}$$

where $\tilde{w}_t = \tilde{y}_t - \tilde{y}_{t-s} - \sum_{j=1}^{s} \tilde{b}_j(\tilde{\phi})\delta_{jt}$, $\tilde{r}_t = \tilde{y}_t - \sum_{j=1}^{s}\{\tilde{a}_j^n(\tilde{\phi}) + \tilde{b}_j(\tilde{\phi})\tau\}\delta_{jt}$, and c_t is an error term.

We note that testing $\theta = 0$ in (2.3) is equivalent to testing $\rho = 1$ in (2.1), see Ahn and Cho (1992). Therefore, test statistics for a seasonal unit root can be easily obtained.

Theorem 1. We let $\tilde{F}$ and $\tilde{t}$ be the conventional F- and t-statistics, respectively, from simple linear regression for testing $\theta = 0$ in (2.3), or equivalently, for testing $\rho = 1$ in Model (2.1). Then

$$\tilde{F} \rightarrow^{D} \frac{s^2}{4\sum_{j=1}^{s}\int_0^1 V_j(r)^2 dr},$$

$$\tilde{t} \rightarrow^{D} -\frac{s}{2\{\sum_{j=1}^{s}\int_0^1 V_j(r)^2 dr\}^{1/2}},$$

where the $V_j(r)$ are independent standard Brownian bridges, that is, $V(r) = W(r) - rW(1)$ with $W(r)$ being a standard Brownian motion on $[0,1]$. Furthermore,

$$T\tilde{\theta} \rightarrow^{D} -\frac{s^2}{2\sum_{j=1}^{s}\int_0^1 V_j(r)^2 dr},$$

where $\tilde{\theta}$ is the least squares regression estimator of θ and T is the total series length, that is, $T = \sum_{j=1}^{s} T_j$. The proof of the theorem is algebraically involved and hence given in Appendix.

Now, comments regarding the results of theorem are in order. First, the asymptotic distributions of the test statistics for testing $H_0: \ \rho = 1$ given in Theorem 1 do not depend on nuisance parameters α_j, β_j, or ϕ_i. Even though $\tilde{F}$ and $\tilde{t}$ are computed the exactly same way as the F- and t-statistics in a simple linear regression, their asymptotic distributions are no longer F- or Student's t- distribution but are functionals of stochastic integrals of Brownian bridges. The asymptotic distributions in the theorem are the same as those in Theorems 1 and 2 of Ahn and Cho (1992) where Model (1.1) was considered. Therefore, for critical values of the test statistics in Theorem 1, percentiles in Ahn and Cho (1992) can be used.

3. NUMERICAL EXAMPLE

In order to illustrate the methods and concepts, we consider quarterly United Kingdom data on the logarithm of consumption expenditures (Y_t) for the period 1955 through 1979. Data are obtained from Hylleberg (1986), and they were briefly analyzed along with the logarithm of personal disposable income for seasonal co-integration features by Ahn and Reinsel (1992). Time series plot of the logarithm of the quarterly consumption expenditure series is displayed in Figure 1. The figure illustrates the nonstationary seasonal character, and therefore can be modeled by (2.1).

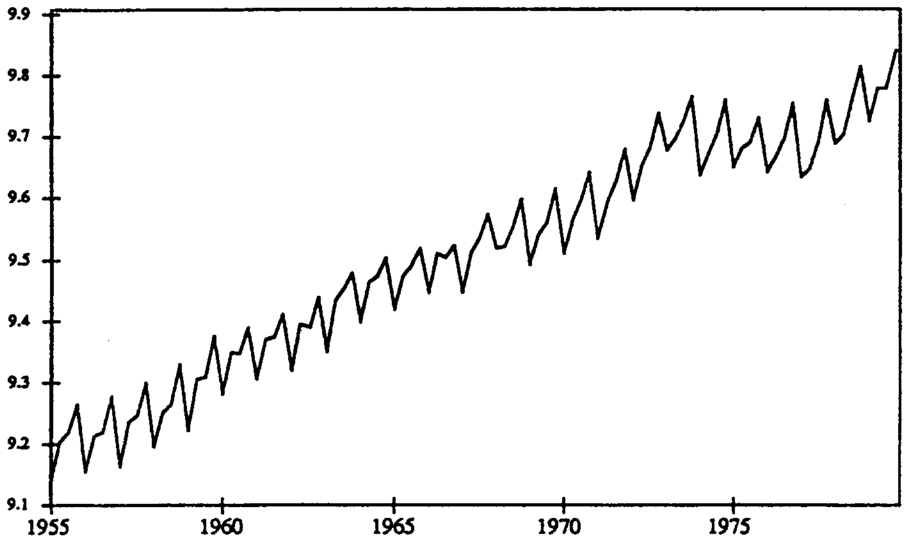

Figure 1. Log of Quarterly U.K. Consumption Expenditure Series, 1955-1979.

Following the procedure in Section 2, we first model the seasonally differenced series $Z_t = Y_t - Y_{t-4}$ as

$$(1-\phi_1 B)(1-\phi_4 B^4)Z_t = \sum_{j=1}^{4} \omega_j \delta_{jt} + e_t,$$

and obtain

$$\tilde{\phi}_1 = 0.77464 \quad \text{and} \quad \tilde{\phi}_4 = -0.40190,$$

using PROC ARIMA of SAS. Next, we obtain $\tilde{y}_t = (1-\tilde{\phi}_1 B)(1-\tilde{\phi}_4 B^4)Y_t$ for $t = 6, 7, \ldots$. (For the sake of convenience we ignored $\tilde{y}_6$ through $\tilde{y}_8$, and used the remaining $\tilde{y}_t$ in the rest of the analysis.) We note that the $(1-B^4)\tilde{y}_t$ behave almost like a white noise series, see the autocorrelation function in Figure 2. Thus application of Ahn and Cho (1992) is appropriate. Using the $\tilde{y}_t$ we obtain

$$\begin{aligned}
\tilde{a}_1^n(\tilde{\phi}) &= 2.76207, & \tilde{b}_1(\tilde{\phi}) &= 0.00935,\\
\tilde{a}_2^n(\tilde{\phi}) &= 2.98413, & \tilde{b}_2(\tilde{\phi}) &= 0.00635,\\
\tilde{a}_3^n(\tilde{\phi}) &= 2.92019, & \tilde{b}_3(\tilde{\phi}) &= 0.00796,\\
\tilde{a}_4^n(\tilde{\phi}) &= 2.98928, & \tilde{b}_4(\tilde{\phi}) &= 0.00796.
\end{aligned}$$

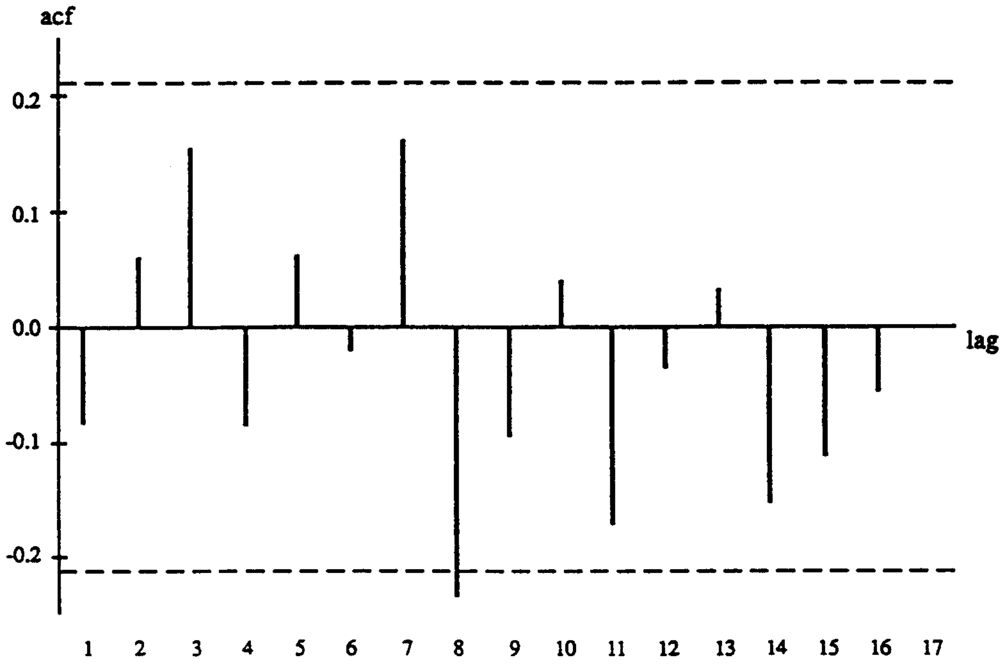

Figure 2. ACF of $(1-B^4)\tilde{y}_t$

Finally, we obtain $\tilde{w}_t$ and $\tilde{r}_t$, and regress $\tilde{w}_t$ on $\tilde{r}_{t-s}$ to obtain $\tilde{F} = 13.675$ without an intercept in the regression model and $\tilde{F} = 15.081$ with an intercept. (We note that even though, asymptotically, the regression model without an intercept term is appropriate, for finite samples the model with an intercept is conjectured to be better.)

Compared with percentiles in Table 1 of Ahn and Cho (1992), we reject the null hypothesis of a seasonal unit root at the significance level of, say, 0.05. One implication of the testing result is that the nonstationary seasonal behavior of the data is not attributable to a stochastic trend but to a deterministic trend, and this is along the line of the aforementioned claims by Barsky and Miron (1989). Another implication is that not all the nonstationary roots of $1 - B^4 = 0$, namely, ± 1 and $\pm i$ are characteristic roots of the autoregressive part of Y_t (or N_t). This is because testing $H_0 : \ \rho = 1$ is equivalent to testing the null hypothesis: all of the nonstationary roots $1, -1$, and $\pm i$ corresponding to frequencies zero, 1/2, and 1/4, respectively, are characteristic roots of the AR part of Y_t or (N_t), and rejection of the null implies that not all of ± 1 and $\pm i$ are characteristic roots. See Hylleberg, Engle, Granger, and Yoo (1990) for a related discussion, especially testing nonstationary roots corresponding to different frequencies separately. We note that Hylleberg et al. (1990) analyzed data similar to the one analyzed in this paper, but their analysis was carried under the assumption of no deterministic trends in the model.

4. POWER STUDY

A small Monte Carlo sampling experiment is conducted in order to investigate the power of the tests proposed in this paper. We consider the following model which has deterministic seasonal trends with a stationary noise term.

$$Y_t = \sum_{j=1}^{4} \beta_j \tau \delta_{jt} + N_t, \tag{4.1}$$

$$(1 - 0.4B)(1 - \rho B^4) N_t = e_t.$$

for $\rho = 0.9,\ 0.95,\ 0.98,\ 0.99,\ 0.995$ and $\beta_1 = 1$, $\beta_2 = 3$, $\beta_3 = 4$, and $\beta_4 = 2$. $\rho = 1$ is also considered in order to study the significance level for finite samples. Samples of series length $T = 400$ (thus $T_j = 100$) were generated using the RNNOA subroutine of IMSL to first generate pseudo-random samples

Table 1. Empirical Powers of the Test Statistics in Theorem 1 for Model (4.1)

Test Statistic	ρ 0.9	0.95	0.98	0.99	0.995	1.0
$\tilde{F}$	.6925	.2330	.0743	.0484	.0470	.0442
$\tilde{t}$	.7366	.2690	.0879	.0602	.0563	.0538
$T\tilde{\theta}$	.7030	.2412	.0781	.0512	.0487	.0467

of the e_t from a standard normal distribution. Based on 10000 replications, empirical powers of the tests were obtained for each of the values of ρ at the level of significance 0.05.

The results of the simulation is summarized in Table 1. It is interesting to note that the empirical power of the regression "t-statistic", $\tilde{t}$ is higher than that of the regression "F-statistic", $\tilde{F}$ or of $T\tilde{\theta}$, and that empirical significance levels of $\tilde{F}$ and $T\tilde{\theta}$ are lower than the nominal level. The empirical powers of the tests are similar to those in Ahn and Cho (1992), where a seasonal model without deterministic trends are considered.

Since, under the null hypothesis Model (2.9) of Dickey, Hasza, and Fuller (1984) has deterministic seasonal components, we also obtained empirical powers of their test statistics: $\hat{\alpha}_{\mu d}$, $\hat{\tau}_{\mu d}$, and $\tilde{\alpha}_{\mu d}$, and found almost no power for Model (4.1). One reason of the poor power of the test statistics in Dickey, Hasza, Fuller (1984) is because under the alternative hypothesis, Model (2.9) of Dickey, Hasza, and Fuller (1984) does not have deterministic seasonal trends, while Model (4.1) has deterministic trends under both hypotheses. Therefore, the tests proposed in this paper can be applied to a general class of nonstationary seasonal models, especially, when there is no *apriori* knowledge about deterministic trends available. The tests in Dickey, Hasza, and Fuller (1984) can be used if *apriori* knowledge about no deterministic trends is readily available. Even though under the null hypothesis Model (2.9) in Dickey, Hasza, and Fuller (1984) accounts for deterministic seasonal trends in model fitting, use of the asymptotic distributions in Dickey, Hasza, and Fuller(1984) is incorrect when there are determinsitc trends, as in Model (4.1), in the underlying process. This is because their asymptotic distributions were obtained under the assumption of no deterministic trends.

APPENDIX

Proof of Theorem 1: As mentioned in Section 2, when ϕ is known, the test statistics are obtained by regressing w_t on r_t, where $w_t = y_t - y_{t-s} - \sum_{j=1}^{s} \tilde{b}_j \delta_{jt}$, $r_t = y_t - \sum_{j=1}^{s}\{\tilde{a}_j^n + \tilde{b}_j \tau\}\delta_{jt}$, $\tilde{b}_j = (y_{(T_j-1)s+j} - y_j)/(T_j - 1)$, and $\tilde{a}_j^n = y_j - \tilde{b}_j$. In this case, using Lemma 1 of Ahn and Cho (1992) we can easily establish that under $H_0: \ \rho = 1$

$$T^{-1}\sum_{t=1}^{T} w_t r_{t-s} \to^D -\sigma^2/2,$$

$$T^{-2}\sum_{t=1}^{T} r_{t-s}^2 \to^D \frac{\sigma^2}{s^2}\sum_{j=1}^{s}\int V_j(r)^2 dr,$$

and the asymptotic distributions of the test statistics follow immediately.

On the other hand, when ϕ is unknown, the test statistics are obtained by regressing $\tilde{w}_t$ on $\tilde{r}_{t-s}$, and for their asymptotic distributions we need to study asymptotic properties of $\sum_{t=1}^{T} \tilde{w}_t \tilde{r}_{t-s}$ and $\sum_{t=1}^{T} \tilde{r}_{t-s}^2$. To this end, we first note that under the null hypothesis the $\tilde{\phi}_i$ obtained from the model of $Z_t = Y_t - Y_{t-s}$ are consistent estimators of the ϕ_i. In fact, by "standard" estimation theory we have $\tilde{\phi}_i - \phi_i = O_p(T^{-1/2})$. Next, using

$$\tilde{b}_j(\tilde{\phi}) - \tilde{b}_j = -\sum_{i=1}^{p}(\tilde{\phi}_i - \phi_i)(Y_{(T_j-1)s+j-i} - Y_{j-i})/(T_j - 1),$$

$$\tilde{y}_{(\tau-1)s+j} - y_{(\tau-1)s+j} = \sum_{i=1}^{p}(\tilde{\phi}_i - \phi_i)Y_{(\tau-1)s+j-i},$$

$$\tilde{a}_j^n(\tilde{\phi}) - \tilde{a}_j^n = \tilde{y}_j - y_j - \{\tilde{b}_j(\tilde{\phi}) - \tilde{b}_j\},$$

we can easily establish

$$T^{-1}\sum_{t=1}^{T} \tilde{w}_t \tilde{r}_{t-s} = T^{-1}\sum_{t=1}^{T} w_t r_{t-s} + O_p(T^{-1/2}),$$

$$T^{-2}\sum_{t=1}^{T} \tilde{r}_{t-s}^2 = T^{-2}\sum_{t=1}^{T} r_{t-s}^2 + O_p(T^{-1/2}),$$

and the rest of the results follows.

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