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Stochastic Comparisons of Order Statistics†

Song-Ho Kim[1]

ABSTRACT

The purpose of this paper is to investigate the properties of order statistics under various stochastic relations. We study the stochastic comparison of order statistics in a single sample. And we consider two sample case too. For example, $F(t) > G(t)$ for $t > 0$ when X and Y are random variables symmetric about 0, with c.d.f.s F and G. Two examples are provided.

KEYWORDS: Peakedness, Stochastic ordering, Dispersive ordering, Exchangable random variables, Symmetric distribution, Student distribution

1. INTRODUCTION

Given a random sample, $X_1, X_2, \cdots, X_n$, we can arrange the X's in ascending order of magnitude and then write $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$, or $(X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)})$. We call $X_{(r)}$ the rth order statistic. If the distribution funtion of X is $F(x)$, then $F_{r:n}(x)$ or $F_r(x)$ denotes the distribution

[1] Department of Statistics, Tae-Shin University, Anyang, 430-015, Korea.
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funtion of the rth order statistics. We investigate some aspects of the stochastic comparisons of the order statistics.

Section 2 defines and reviews various stochastic order ralations, including stochastic ordering, peakedness ordering, r-ordering, dispersive ordering, star-shaped ordering, r-ordering, dispersive ordering, and likelihood ratio ordering. An extensive review of the literature is given in Kim(1988).

In Section 3, we first investigate the stochastic comparison of order stochastic from exchangeable random variables in a single sample. And using various stochastic orderings, we will investigate the comparisons of the reliabilities and expected values of the order stochastic for two populations. For example, corresponding to the stochastic ordering between X and Y, we consider the case

$$F(t) \geq (<) G(t) \text{ respectively for } t \geq (<) 0 \tag{1.1}$$

when X and Y are random variables symmetric about 0, with c.d.f.s F and G. We consider the case, $F <_r G$ (see Definition 2.5) with $f(0) \geq g(0) > 0$, which is a stronger assumption than (1.1). Also examples are provided in each case.

2. STOCHASTIC ORDER RELATIONS

We introduce some stochastic order relations between random variables. First we consider the concept of one random variable being stochastically larger than another.

Definition 2.1. The random variable X is stochastically larger than Y, written $X \leq (st) Y$.

$$\text{if } \; P\{Y > t\} \leq P\{X > t\} \text{ for all } t. \tag{2.1}$$

If X and Y are nonnegative random variables, then (2.1) implies $E[X] \geq E[Y]$.

Definition 2.2. [Birnbaum (1948)] Y is more peaked than X,

$$\text{if } P\{|Y| > t\} \leq P\{|X| > t\} \text{ for all } t > 0. \tag{2.2}$$

If $X(Y)$ has symmetric random variable with distribution function $F(G)$ about 0, (2.2) is equivalent to $G(t) \geq F(t)$ for all $t > 0$.

In order to compare relative skewness, Van Zwet(1964) defined convex ordering.

Definiton 2.3. [Van Zwet(1964)] Let F and G be continuous distributions, with G strictly increasing on its support, an interval. Then F is convex with respect to G(written $F <_C G$) if $G^{-1}F(x) - x$ is a convex function of x on the support of F.

In reliability theory, if F is a life distribution, $G(x) = 1 - e^{-x}$, and $F <_C G$, then F is said to have increasing failure rate (IFR). Van Zwet(1964) also introduced another orderings, s-ordering, that is restricted to the class of symmetric distributions. The main purpose of this ordering is the comparison of relative heaviness of tail among symmetric distributions.

Barlow and Proschan(1966) introduced star-shaped ordering, a weaker ordering than convex ordering.

Definition 2.4. [Barlow and Proschan (1966)] Let F and G be continuous distibutions, with G strictly increasing on its support, and $F(0) = G(0) = 0$. Then, F is star-shaped with respect to G (written $F <_* G$) if $G^{-1}F(x)/x$ is increasing for $x > 0$.

Lawrence(1975) introduced r-ordering, a weaker ordering than s-ordering : $F <_r G$ iff $G^{-1}F(x)/x \quad \uparrow (\downarrow)$ w.r.t. $x > (<)0$, and $F(0) = G(0) = \frac{1}{2}$.

In order to compare relative dispersiveness, Lewis and Thompson (1981) defined dispersive ordering.

Definition 2.5. [Lewis and Thompson(1981)] If any two quantiles of G are more widely separated than the corresponding quantiles of F, Then $F <_{disp} G$ (i.e. $F^{-1}(\alpha) - F^{-1}(\beta) \leq G^{-1}(\alpha) - G^{-1}(\beta)$ for any $0 < \alpha < \beta < 1$).

Remark 2.5. Deshpande and Kochar(1983) and Shaked(1982) discussed on the characterization of dispersive ordering: Shaked(1982) showed that when F and G are two distribution functions which are strictly increasing and continuous on their support $[0,\infty)$, then $F <_{disp} G$ iff (a) $F(x) - G(x) \geq 0$ for all $x \in [0,\infty)$ and (b) for every $c > 0$, the distribution functions of $X + c$ and Y cross at most once and if there is a sign change, $F_{X+c} - G_Y$ changes sign from $-$ to $+$.

Definition 2.6. [Karlin (1968)] Let X and Y denote continuous random variables having respective densities f and g. We say X is larger than Y in the sense of likelihood ratio, and write $X \geq_{LR} Y$ if

$$\frac{f(x)}{g(x)} \leq \frac{f(y)}{g(y)} \text{ for all } x < y.$$

Remark 2.6. It can be shown that $X \geq_{LR} Y$ implies $X \geq_{st} Y$. Also Ross(1983, P.260) showed that $X \geq_{LR} Y \Rightarrow \dfrac{f_X(t)}{1 - F_X(t)} \leq \dfrac{f_Y(t)}{1 - F_Y(t)}$ for all $t \geq 0$, provided X and Y are nonnegative random variables. (Note $\dfrac{f(t)}{1 - F(t)}$ is the failure rate at time t.)

3. STOCHASTIC COMPARISONS OF ORDER STATISTICS

Firstly, we will discuss the comparison of order statistics from one sample. We need the following additional notation:

$X^*_{r:n}$, the rth order statistic in any subset of n exchangeable variates drawn from a larger set $X_1, X_2, \cdots, X_{n'}, (n' > n)$, each with marginal c.d.f. $F(x)$.

Thus, the joint distribution of $X^*_{r:n}$ and $X^*_{s:n}$ is not the same as that of $X_{r:n}$ and $X_{s:n}$.

Theorem 3.1. Let $X_1, X_2, \cdots, X_n$ be exchangeable ramdom variables. Then, for $n_1 < n_2 < \cdots < n_k \leq n$,

$$X^*_{1:n} \leq_{st} X^*_{2:n} \leq_{st} \cdots \leq_{st} X^*_{n:n}, \tag{3.1}$$

$$X^*_{i:n_k} \leq_{st} X^*_{i:n_{k-1}} \leq_{st} \cdots \leq_{st} X^*_{i:n_1}, \tag{3.2}$$

and

$$X^*_{n_1-i:n_1} \leq_{st} X^*_{n_2-i:n_2} \leq_{st} \cdots \leq_{st} X^*_{n_k-i:n_k}. \tag{3.3}$$

Proof. Since $X_1, X_2, \cdots, X_n$ are exchangeable, $X^*_{i:n} =_{st} X_{i:n}$ for $i = 1, 2, \cdots, n$. But $P\{X_{i:n} \leq X_{j:n}\} = 1$ for $i < j$. Hence, (3.1) follows by coupling (Ross(1983, P. 255)). Inequalities (3.2) and (3.3) follow similarly.

Theorem 3.2. Let $X_1, X_2, \cdots, X_n$ be a random sample from a continuous population with c.d.f. F. Let $n_1 < n_2$. Then,

$$X_{r:n_1} \geq_{LR} X_{s:n_2} \quad \text{for } r \geq s$$

and

$$X_{r:n_1} \leq_{LR} X_{s:n_2} \quad \text{for } r \leq s - n_2 + n_1$$

where $1 \leq r \leq n_1$ and $1 \leq s \leq n_2$.

Since $X \geq_{LR} Y$ implies $X \geq_{st} Y$, we can replace likelihood ordering by stochastic ordering in Theorem 3.1 if $X_1, X_2, \cdots, X_n$ are a random sample.

Theorem 3.3. [Madreimov and Petunin (1985)] Let $X_1, X_2, \cdots, X_n$ be a random sample from a population for which its distribution is symmetric and unimodal continuous random variable with $F(x)$, having finite expected value. Then,

$$[X_{r+1:n} - X_{r:n}] \geq_{st} [X_{r:n} - X_{r-1:n}] \quad \text{for } \frac{(n+1)}{2} \leq r \leq n \tag{3.4}$$

and

$$[X_{r+1:n} - X_{r:n}] \leq_{st} [X_{r:n} - X_{r-1:n}] \quad \text{for } 1 \leq r \leq \frac{(n+1)}{2}. \tag{3.5}$$

Remark. (3.4) and (3.5) imply

$$E[X_{r+1:n} - X_{r:n}] \geq E[X_{r:n} - X_{r-1:n}] \quad \text{for } \frac{(n+1)}{2} \leq r \leq n$$

and

$$E[X_{r+1:n} - X_{r:n}] \leq E[X_{r:n} - X_{r-1:n}] \quad \text{for } 1 \leq r \leq \frac{(n+1)}{2}.$$

Secondly, we will discuss the stochastic comparisons of order ststistics from two sample. Let $X_1, X_2, \cdots, X_n (Y_1, Y_2, \cdots, Y_n)$ be random samples with c.d.f.s $F(G)$. Since

$$E[Y_{r:n}] - E[X_{r:n}] = \int_{-\infty}^{\infty} n[G^{-1}F(x) - x] \binom{n-1}{r-1} F^{r-1}(x)(1-F(x))^{n-r} dF(x) \tag{3.6}$$

$$= \int_{-\infty}^{\infty} n[G^{-1}F(x) - x] dF_{r:n}(X), \tag{3.7}$$

for the comparison of expected values of order statistics from two different populations, the function $G^{-1}F(x) - x$ play a prominent role. Doksum (1974) calls the function $G^{-1}F(x) - x$, the shift function. Note that the convexity of $G^{-1}F(x) - x$ is equivalent to $F <_C G$.

Similarly, for nonnegative random variables,

$$\frac{G^{-1}F(x) - x}{x} \uparrow \quad \text{w.r.t.} x \Leftrightarrow F <_* G.$$

Also, since $\binom{n-1}{r-1} F^{r-1}(x)(1 - F(x))^{n-r}$ is totally positive function in r and x, (3.6) shows that the number of sign changes in $E[Y_{r:n}] - E[X_{r:n}]$ with r is no greater than the number of sign changes in $G^{-1}F(x) - x$ as $x : -\infty \to \infty$, by the variation diminishing property of totally positive functions (Karlin (1968, P.21); Boland and Proschan (1986)). For example, if no sign change occurs (i.e., $G^{-1}F(x) - x \geq 0$ for all x), then $E[Y_{r:n}] \geq E[X_{r:n}]$ for $r = 1, 2, \cdots, n$.

Theorem 3.4. [Oja (1981)] Let F and G be absolutely continuous distribution functions from random variables X and Y. If $F <_{disp} G$, then

$$[X_{s:n} - X_{r:n}] \leq_{st} [Y_{s:n} - X_{r:n}] \text{ for any } 1 \leq r < s \leq n.$$

Remark 3.4.1. This was shown by the characterization of dispersive ordering:

$$\begin{aligned} G^{-1}F(x) - x \uparrow \quad \text{w.r.t.} x \;&\Leftrightarrow\; G^{-1}F(x) - x \leq G^{-1}F(y) - y \quad (\text{for any } \; x < y) \\ &\Leftrightarrow\; y - x \leq G^{-1}F(y) - G^{-1}F(x) \\ &\Leftrightarrow\; F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha) \;\text{ with} \\ &\; F(x) = \alpha \;\text{ and }\; F(y) = \beta \ni 0 < \alpha < \beta < 1, \end{aligned}$$

$G^{-1}F(x) - x \uparrow$ w.r.t. $x \Leftrightarrow F <_{disp} G$ (i.e., $\frac{d}{dt}[G^{-1}F(t) - t] > 0. \Leftrightarrow F <_{disp} G$). Deshpande and Kochar(1983) showed this.

Remark 3.4.2. If any two quantiles of G are more widely separated than the corresponding quantiles of F, then spacings of the $Y_{i:n}$ are stochastically larger than the corresponding spacings of the $X_{i:n}$.

Intuitively, the expected values of order ststistics depend on skewedness, and peakedness or heaviness of tail. Let X and Y be random variables symmetric about zero with c.d.f.s F and G. If X is more peaked than Y, then this is the one sign change of shift function, $G^{-1}F(t) - t \geq (<)0$ w.r.t. $t \geq (<)0$.

Theorem 3.5. Let X and Y be random variables symmetric about 0, with c.d.f.s F and G. If X is more peaked than Y (i.e. $G(x) \leq F(x)$ for all $x > 0$),then

$$E[Y_{r:n}] \geq E[X_{r:n}] \quad \text{for } \frac{(n+1)}{2} \leq r \leq n.$$

Proof. From David(1981, P.38),

$$E[Y_{r:n}] \geq E[X_{r:n}] = \int_0^\infty [G_{n-r+1:n}(x) - F_{n-r+1}(x) - G_{r:n}(x) + F_{r:n}(x)]dx.$$

Now it suffices to show that

$$F_{r:n}(x) - G_{r:n}(x) \geq F_{n-r+1:n}(x) - G_{n-r+1:n}(x) \text{ for } r \geq \frac{1}{2}(n+1) \text{ and}$$

$x > 0$.

$$\begin{aligned}\text{LHS} &= I_{F(x)}(r, n-r+1) - I_{G(x)}(r, n-r+1),\\ &= \int_{G(x)}^{F(x)} \frac{t^{r-1}(1-t)^{n-r}}{B(r, n-r+1)} dt,\end{aligned}$$

where I denotes on Incomplete Beta function.

Similarly,

$$\text{RHS} = \int_{G(x)}^{F(x)} \frac{t^{n-r}(1-t)^{r-1}}{B(r, n-r+1)} dt.$$

Since $\frac{t^{r-1}(1-t)^{n-r}}{t^{n-r}(i-t)^{r-1}} = (\frac{t}{1-t})^{2r-1-n} \geq 1$ for $r \geq \frac{1}{2}(n+1)$ and $t \geq \frac{1}{2}$, we have

$$E[Y_{r:n}] \geq E[X_{r:n}] \text{ for } \frac{1}{2}(n+1) \leq r \leq n.$$

Remark 3.5.1. Since

$$F_{r:n}(x) \geq (<) G_{r:n}(x) \text{ iff } F(x) \geq (<) G(x) \quad r = 1, 2, \cdots, n$$

i.e., iff $x \geq (<) 0$.

It follows that neither of $X_{r:n}$, nor of $Y_{r:n}$ is stochastically larger than the other.

Corollary 3.5. For $\frac{1}{2}(n+1) \leq s \leq n$ and $1 \leq r \leq \frac{1}{2}(n+1)$, $E[Y_{s:n} - Y_{r:n}] \geq E[X_{s:n} - X_{r:n}]$ under the same assumptions as Theorem 3.5.

The proof follows immediately from symmetry consideration.

Example 3.5. The T distribution with m degrees of freedom is symmetrically distributed about 0 with differentiable and unimodal p.d.f.,

$$f_m(t) = \frac{\Gamma(\frac{m+1}{2})}{\sqrt{m\pi}\Gamma(\frac{m}{2})}(1 + \frac{t^2}{m})^{-\frac{(m+1)}{2}} \quad -\infty < t < \infty.$$

If $F_{m+1}(x) \geq F_m(x)$ for all $x > 0$ (i.e. T_{m+1} is more peaker than T_m), then by Theorem 3.3, Theorem 3.5 and Corollary 3.5, we have

$$[T_{m(s+1)}-T_{m(s)}] \geq_{st} [T_{m(s)}-T_{m(s-1)}] \text{ for } \frac{1}{2}(n+1) \leq s \leq n, \quad (3.8)$$

$$E[T_{m(s)}] \geq E[T_{m+1(s)}] \text{ for } \frac{1}{2}(n+1) \leq s \leq n, \quad (3.9)$$

and

$$E[T_{m+1(s)} - T_{m+1(r)}] \leq E[T_{m(s)} - T_{m(r)}]$$

for $1 \leq r \leq \frac{1}{2}(n+1)$ and $\frac{1}{2}(n+1) \leq s \leq n$. (3.10)

Remark 3.5.2. (3.8) imply

$$E[T_{m(s+1)} - T_{m(s)}] \geq E[T_{m(s)} - T_{m(s-1)}] \text{ for } \frac{1}{2}(n+1) \leq s \leq n. \quad (3.11)$$

Before we prove $F_{m+1}(x) \geq F_m(x)$ for all $x > 0$, we need the following two lemmas.

Lemma 3.5.1. $f_{m+1}(0) > f_m(0)$ for any $m = 1, 2, \cdots$.

Proof. $f_m(0) = \dfrac{\Gamma(\frac{m+1}{2})}{\sqrt{m\pi}\Gamma(\frac{m}{2})} = \dfrac{1}{\sqrt{m}B(\frac{1}{2},\frac{m}{2})}$, where B(a,b) is a Beta function. Similarly, $f_{m+1}(0) = \dfrac{1}{\sqrt{m+1}B(\frac{1}{2},\frac{m+1}{2})}$. It suffices to prove that $\dfrac{1}{f_m(0)} - \dfrac{1}{f_{m+1}(0)} > 0$ (i.e., $\sqrt{m}B(\frac{1}{2},\frac{m}{2}) - \sqrt{m+1}B(\frac{1}{2},\frac{m+1}{2}) > 0$).

$$\sqrt{m}B(\frac{1}{2},\frac{m}{2}) - \sqrt{m+1}B(\frac{1}{2},\frac{m+1}{2}) = \int_0^1 X^{\frac{1}{2}-1}(1-X)^{\frac{m}{2}-1}[\sqrt{m}-\sqrt{m+1}(1-x)^{\frac{1}{2}}]dx.$$

Since $\sqrt{m} - \sqrt{m+1}(1-X)^{\frac{1}{2}}$ is strictly convex on (0,1), by Jensen's inequality the integral is greater than

$$\sqrt{m} - \sqrt{m+1}[1 - E(x)]^{\frac{1}{2}} = 0 \ (\textit{by } E(x) = \frac{1}{m+1}).$$

Hence, $f_{m+1}(0) > f_m(0)$ for any $m = 1, 2, \cdots$.

Lemma 3.5.2. $f_{m+1}(1) > f_m(1)$ for any $m = 1, 2, \cdots$.

Similarly, it can be shown easily.

Claim. $F_{m+1}(x) \geq F_m(x)$ for all $x \geq 0$.

Proof.

$$\frac{d}{dt}\left[\frac{f_m(t)}{f_{m+1}(t)}\right] = c\frac{t(t^2-1)(m+t^2)^{\frac{1}{2}(m-1)}(m+t^2+1)^{\frac{1}{2}m}}{[m+t^2]^{(m+1)}}$$

where $c = \dfrac{\dfrac{\Gamma[\frac{m+1}{2}]}{\sqrt{m\pi}\Gamma[\frac{m}{2}]}m^{\frac{1}{2}(m+1)}}{\dfrac{\Gamma[\frac{m+2}{2}]}{\sqrt{(m+1)\pi}\Gamma[\frac{m+1}{2}]}(m+1)^{\frac{1}{2}(m+2)}}$.

Hence

$$\frac{f_m(t)}{f_{m+1}(t)} \uparrow \text{ w.r.t. } t \text{ on } t > 1 \tag{3.12}$$

and

$$\frac{f_m(t)}{f_{m+1}(t)} \downarrow \text{ w.r.t. } t \text{ on } 0 < t < 1. \tag{3.13}$$

From Lemma 3.5.1, Lemma 3.5.2, (3.12) and (3.13), we have $F_{m+1}(x) \geq F_m(x)$ for $x \geq 0$ and for $m = 1, 2, \cdots$.

Using Tiku and Kumra(1985), we can see (3.9), (3.10) and (3.11) illustrated in Table 3.5.

Table 3.5. Expected values of order statistics of t-distributions from $m = 3$ degrees of freedom to $m = 19$ and standard normal distribution in sample size, $10(n = 10)$(Tiku and Kumra(1985)).

m	$E[T_{6:10}]$	$E[T_{7:10}]$	$E[T_{8:10}]$	$E[T_{9:10}]$	$E[T_{10:10}]$
3	0.1395858	0.4342024	0.7862594	1.2980512	2.5283165
4	0.1350104	0.4181056	0.7490249	1.2073913	2.5283165
5	0.1323781	0.4089456	0.7283123	1.5290206	2.0028543
6	0.1306693	0.4030378	0.7151333	1.1289959	1.9046147
7	0.1294709	0.3989131	0.7060142	1.1085582	1.8405509
8	0.1285844	0.3958706	0.6993311	1.0937537	1.7955165
9	0.1279020	0.3935343	0.6942235	1.0825376	1.7621470
10	0.1273606	0.3916840	0.6901936	1.0737472	1.7621470
11	0.1269207	0.3901824	0.6869328	1.0666731	1.7160262
12	0.1265561	0.3889395	0.6842404	1.0608576	1.6994301
13	0.1262493	0.3878938	0.6819797	1.0559925	1.6856720
14	0.1259869	0.3870019	0.6800546	1.0518625	1.6740820
15	0.1257604	0.3855608	0.6769510	1.0452289	1.6641855
16	0.1255630	0.3855608	0.6769510	1.0452289	1.6556368
17	0.1253892	0.3849705	0.6756820	1.0425251	1.6484783
18	0.1252351	0.3844473	0.6745581	1.0401351	1.6416140
19	0.1250975	0.3839803	0.6735561	1.0380073	1.6357924
∞	0.1226678	0.3858647	0.6560591	1.0013570	1.5387527

Theorem 3.6. Let X and Y be random variables symmetric about 0, with c.d.f.s F and G. Then, if $F \leq_r G$ and $f(0) \geq g(0) > 0$, then $[X_{s:n} - X_{r:n}] \leq_{st} [Y_{s:n} - Y_{r:n}]$ for any $1 \leq r < s \leq n$.

Proof. Deshpande and Kochar(1983) showed: Let F and G be absolutely continuous such that $F(0) = G(0) = 0$ and let the corresponding density functions be such that $f(0) \geq g(0) > 0$. Then $F <_* G$ implies $F <_{disp} G$.

Let $X(Y)$ be a symmetric random variable with density $f(g)$ about 0. From the above argument, if $f(0) \geq g(0) > 0$ and $\dfrac{G^{-1}F(x)}{x} \uparrow$ w.r.t. $x > 0$, then

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha) \text{ for any } \frac{1}{2} < \alpha < \beta < 1. \qquad (3.14)$$

By symmetric consideration, (3.14) implies $F <_{disp} G$. Simply applying Theorem 3.4, we have $[X_{s:n} - X_{r:n}] \leq_{st} [Y_{s:n} - X_{r:n}]$ for any $1 \leq r < s \leq n$.

Example 3.6. Van Zwet(1964) showed that any symmetric U-shaped density $<_s$ uniform $<_s$ normal $<_s$ logistic $<_s$ Laplace $<_s$ Cauchy (see Van Zwet (1964, p. 70-73). Hence, if we change scale in order to meet the assumption, $f(0) \geq g(0) > 0$, then U-shaped density $<_{disp}$ uniform $<_{disp}$ normal $<_{disp}$ logistic $<_{disp}$ Laplace $<_{disp}$ Cauchy. (Note : By Theorem 3.6, these relationships imply the stochastic ordering among the spacings.)

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