不連續 갤러킨 方法에 依한 常微分方程式의 有限要素解析

Finite Element Solution of Ordinary Differential Equation by the Discontinuous Galerkin Method

金 致 京*
Kim, Chi Kyung

요 약

時間變數데 對하여 不連續性을 주는 時間不連續 Galerkin 方法을 有限要素法으로 解析하였다. 이 方法은 微分方程式 觀點에서 지금까지 要素間에 連續性을 준 一般的 有限要素法과 다르게 任意의 時間要素를 選擇, 每 時間段階에서 要素境界에 不連續을 許諾함으로서 解의 正確性을 높이고 無條件의 安定을 주는 常微分 方程式의 解法인 것이다.

Abstract

A time-discontinuous Galerkin method based upon using a finite element formulation in time has evolved. This method, working from the differential equation viewpoint, is different from those which have been generally used. They admit discontinuities with respect to the time variable at each time step. In particular, the elements can be chosen arbitrarily at each time step with no connection with the elements corresponding to the previous step. Interpolation functions and weighting functions are taken to be discontinuous across inter-element boundaries. These methods lead to a unconditional stable higher-order accurate ordinary differential equation solver.

1. Introduction

Most finite element procedures for time-dependent problems are based upon a space discretization which is independent of time. This method reduces the governing partial differential equations to a system of ordinary differential equations in time. The differential equations are in turn discretized by traditional finite difference methods to provide algebraic equations. Procedures of this kind are, at least conceptually, easy to perform. However, they can be expensive if steep gradients occur in the solution ; stability must be controlled, and the global error control can be troublesome. Moreover, these methods are less approriate for some-dependent problems that include free surface boundary value problems.

* 정회원, 시립인천대학교 산업안전공학과

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Recently, the time-discontinuous Galerkin method based upon using a finite element formulation in time has evolved. This method, working from the previously discussed. They admit discontinuities with respect to the time variable at each time step. In particular, the elements can be chosen arbitrarily at each time step with no connection with the elements corresponding to the previous step. Interpolation functions and weighting functions are taken to be discontinuous across inter-element boundaries. These methods have performed well in practice, and a comprehension set of mathematical results has been established by Bonnerot and Jamet [1], Delfour et al [2], Hughes et al [4-6], Jamet [3], and Lesaint and Raviart [7].

2. A Time-Discontinuous Galerkin Method for Ordinary Differential Equations

2.1 Preliminaries

Solutions to the transient parabolic type problems are similar to the solution of a system of first-order linear differential equations of the form

$$\dot{u} + \alpha u = f \quad \text{for} \quad t \geq 0 \tag{2.1}$$

$$u(0) = u_o \tag{2.2}$$

where $u=u(t)$ is the dependent variable, $\dot{u}$ denotes differentiation of u with respect to time, α, u_o are given values and $f(t)$ is a specified function.

Consider a partition of the time domain, $I=[0,T]$, having the form to $< t_1 < t_2 < \cdots < t_N=T$. Let $\{t_n : 0 \leq n \leq N\}$ be a finite sequence of real numbers with $t_o=0$, $t_n < t_{n+1}$ and denote by $I_n=[t_n, t_{n+1}]$ the n-th time interval. For a give non-negative integer r the space of interpolation functions is.

$$S^h=\{u^h : I \rightarrow U^h \,|\, I_n \in P_r[I_n],\ n=1,2,3,\cdots,N\} \tag{2.3}$$

where

$$P_r[I_n]=\{u^h : I_n \rightarrow U^h : u^h(t) = \sum_{1}^{r} U_i t^i \text{ with } u_i \in U^h\} \tag{2.4}$$

U^h is the finite-dimensional subspace of continuous piecewise polynomial functions of degree r. S^h is the space of functions on I with values in U^h that on each time interval I_n vary as polynomials of degree at most r. P_r is the set of polynomials on I_n of degree less than or equal to r. The superscript h stands for maximum time interval in the domain. To account for this, a temporal jump operator of u is introduced with the notation.

$$[u(t_n)]=u(t_n^+)-u(t_n^-) \tag{2.5}$$

where

$$u(t_n^+)=\lim_{\varepsilon \rightarrow 0+} u(t_n^-)=\lim_{\varepsilon \rightarrow 0-} u(t_n-\varepsilon) \tag{2.6}$$

and we will also use the following notations.

$$u_n^+=u(t_n^+),\ u_n^-=u(t_n^-) \tag{2.7}$$

2.2 Weighted Residual Formulation

Using the usual weighted residual approach to develop an integral statement equivalent to Equation (2.1), the time-discontinuous Galerkin approximation to u^h on each interval I_n can now be formulated as

$$\int_{t_n^-}^{t_{n+1}^-} (\dot{u}^h+\alpha\, u^h)\, w^h\, dt=\int_{t_n^-}^{t_{n+1}^-} f\, w^h\, dt \tag{2.8}$$

which can also be written as

$$\int_{t_n^-}^{t_n^+} \dot{u}^h w^h dt + \alpha \int_{t_n^-}^{t_n^+} u^h w^h dt + \int_{t_n^+}^{t_{n+1}^-} (\dot{u}^h + \alpha u^h) w^h dt = \int_{t_n^-}^{t_{n+1}^-} f w^h dt \quad (2.9)$$

This equation is decomposed into discontinuous and continuous time intervals. To accomplish integration of the first and the second term pertaining to discontinuous time interval in Equation(2.9), unit impulse function and average value are introduced. The mathematical definition of the unit impulse function is

$$\delta(t-t_n)=0 \qquad \text{for } t \neq t_n \quad (2.10)$$

$$\int_0^\infty \delta(t-t_n)dt=1 \text{ for } 0 < t_n < \infty \quad (2.11)$$

The value of the approximation at t_n, a point of discontinuity in the approximating polynomial u(.), is given by

$$(u(t_n^-)+u(t_n^+))/2 \quad (2.12)$$

This is an average across the jump.

On the other hand, the derivative of $u(t^n)$ with respect to time is defined as

$$\dot{u}(t^n)=\lim_{\varepsilon\to 0}\frac{u(t_n+\varepsilon/2)-u(t_n-\varepsilon/2)}{\varepsilon} \quad (2.13)$$

$$=(u(t_n^+)-u(t_n^-))\,\delta(t-t_n) \quad (2.14)$$

$$=[u(t_n)]\,\delta(t-t_n) \quad (2.15)$$

Substituting of delta function and average mean value into Equation(2.9) leads to

$$\int_{t_n^+}^{t_{n+1}^-} (-u^h \dot{w}^h+\alpha u^h w^h)\, dt + u^h(t_{n+1}^-)\, w^h(t_{n+1}^-) - u^h(t_n^-)\, w^h(t_n^+) = \int_{t_n^-}^{t_{n+1}^-} f w^h dt \quad (2.16)$$

with the initial condition

$$u^h(t_o^-)=u_o \quad (2.17)$$

The integral in Equation(2.16) resulting from integration-by-parts of the time flux constitutes the discontinuous Galerkin formulation. The last term on the left-hand side of Equation(2.16) from the jump condition

$$w^h(t_n^+)\,[u(t_n)]$$

weakly enforces the initial conditions for each time interval.

2.3 Finite Element Formulation

The specification of the approximation of the dependent variable is accomplished in two steps, the approximation of the variable and the specification of the interpolation function. For the sake of simplicity we define the temporal domain to be a uniform mesh with spacing Δt : $t_n=t+n(\Delta t)$, $0 \le n \le N$ in order that we may approximate u(t) by a function $u^h(t)$ which, on each partitioned subinterval $[t_n, t_{n+1}]$ reduces to a piecewise q-th degree polynomial. We can write the finite element functions for the n-th mesh

$$u^h(t)=N_{n+0}(t)\, u_{n+1}^h + N_{n+1}(t)\, u_{n+1}^h \quad \text{for} \quad t \in I_n \quad (2.18)$$

where u_{n+0}^h are the nodal values of u^h at nodal points t_n^+, u_{n+1}^h at t_{n+1}^- and $N_{n+1}(t)$ in this approximation.

Consider now a time element of length Δt with u^h taking on nodal values u_{n+0} and u_{n+1} as shown in Figure 1. These interpolation functions are assumed to be piecewise linear and are defined as

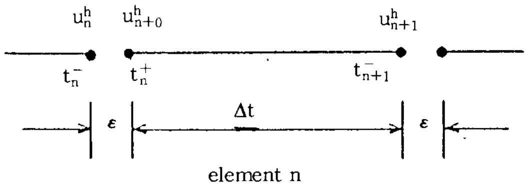

Figure 1. Element in the time domain

$$N_{n+0}(t)=\frac{t_{n+1}-t}{\Delta t},\ N_{n+1}(t)\frac{t-t_n}{\Delta t} \qquad (2.19)$$

Insertion of the approximation for the function into the variational Equation (2.8) can be written as

$$\int_{t_n^+}^{t_{n+1}^-}[-(N_n u_{n+0}^h+N_{n+1}u_{n+1}^h)\dot{w}_\beta^h$$
$$+\alpha(N_n u_{n+0}^h+N_{n+1}u_{n+1}^h)w_\beta^h]\,dt$$
$$+u^h(t_{n+1}^-)\,w_\beta^h(t_{n+1}^-)-u^h(t_n^-)\,w_\beta^h(t_n^+)$$
$$=\int_{t_n^+}^{t_{n+1}^-}[f\,w_\beta^h]dt \text{ for } \beta=n+0,\ n+1 \qquad (2.20)$$

Using $w_\beta^h=N_\beta$, and substituting the above functions of equation into Equation (2.19), we get a two-point in time coupled difference stencil :

$$[\frac{1}{2}+\frac{1}{3}\alpha\,\Delta t]u_{n+0}^h+[\frac{1}{2}+\frac{1}{6}\alpha\,\Delta t] \qquad (2.21)$$
$$u_{n+1}^h-u_n^h=\frac{1}{2}\Delta t\,f_{n+0} \quad \text{for} \quad \beta=n+0$$
$$[\frac{1}{2}+\frac{1}{6}\alpha\,\Delta t]u_{n+0}^h+[\frac{1}{2}+\frac{1}{3}\alpha\,\Delta t]u_{n+1}^h$$
$$-u_n^h=\frac{1}{2}\Delta t\,f_{n+0} \quad \text{for} \quad \beta=n+1$$

The above equation is a recursive relationship, containing unknown values linking two times n+0 and n+1 spaced Δt apart. Because the initial values of the unknowns are specified by Equation (2.2), unknown values at successive times can be calculated. These linear algebraic equations are solved recursively, with iteration at each time step if necessary.

3. Numerical Implementation and Discussions

To illustrate the formulation of the method and its efficiency the solution of a simple problem is presented. For the purpose of comparing the numerical results of the present method, we have solved a single variable equation with $\alpha=1$ and $u_0=1$. The exact solution for this problem is

$$u(t)=e^{-t} \qquad (2.22)$$

Table 1 shows numerical results for each of other methods with $\Delta t=0.1$ compared to the exact solution. These results are plotted in Figure 2. All four methods give close agreement to the exact solution for a short period of time. The explicit method consistently underestimate the solution while the implicit results are always larger than the exact solution to the point that at t=2 seconds. They are under and over estimated the solution by almost 10% respectively. The continuous finite element results oscillate about the exact response but are still within 2% through 2 seconds solutions time. The discontinuous finite element method produces results that are much more accurate with an estimated response that agrees to four decimal places for the full 3 seconds. The plotted results in Figure 2 show how the methods compare to the exact solution with Δt =0.1 sec.

To study the convergence of the different methods, the results for time steps of 0.2 sec, 0.3 sec, 0.4 sec and 0.5 sec were obtained. For Δt=0.4 seconds, the results of the four methods are compared in Figure 3. All methods except the continuous finite element method eventually converge to the exact sol-

Table 1. Comparison of the calculated solution using various time-stepping schemes with the exact solution with $\Delta t=0.1$ sec

Time	Exact	Explicit	Implicit	Con.F.E.M	Dis.F.E.M
0.0	0.10000E+01	0.10000E+01	0.10000E+01	0.10000E+01	0.10000E+01
0.1	0.90484E+00	0.90000E+00	0.90909E+00	0.90484E+00	0.90484E+00
0.2	0.81873E+00	0.81000E+00	0.82645E+00	0.73928E+00	0.81873E+00
0.3	0.74082E+00	0.72900E+00	0.75132E+00	0.73928E+00	0.74082E+00
0.4	0.67032E+00	0.65610E+00	0.68301E+00	0.67071E+00	0.67032E+00
0.5	0.60653E+00	0.59049E+00	0.62092E+00	0.60504E+00	0.60653E+00
0.6	0.54881E+00	0.53144E+00	0.56448E+00	0.54937E+00	0.54881E+00
0.7	0.49659E+00	0.47830E+00	0.51316E+00	0.49512E+00	0.49658E+00
0.8	0.44933E+00	0.43047E+00	0.46651E+00	0.45004E+00	0.44932E+00
0.9	0.40657E+00	0.38742E+00	0.42410E+00	0.40511E+00	0.40657E+00
1.0	0.36788E+00	0.34868E+00	0.38555E+00	0.36873E+00	0.36788E+00
1.1	0.33287E+00	0.31381E+00	0.35050E+00	0.33139E+00	0.33287E+00
1.2	0.30119E+00	0.28243E+00	0.31863E+00	0.30218E+00	0.30119E+00
1.3	0.27253E+00	0.25419E+00	0.28967E+00	0.27102E+00	0.27253E+00
1.4	0.24660E+00	0.22877E+00	0.26333E+00	0.24772E+00	0.24659E+00
1.5	0.22313E+00	0.20589E+00	0.23939E+00	0.22157E+00	0.22313E+00
1.6	0.20190E+00	0.18530E+00	0.21763E+00	0.20314E+00	0.20189E+00
1.7	0.18268E+00	0.16677E+00	0.19785E+00	0.18107E+00	0.18268E+00
1.8	0.16530E+00	0.15009E+00	0.17986E+00	0.16667E+00	0.16530E+00
1.9	0.14957E+00	0.13508E+00	0.16351E+00	0.14788E+00	0.14957E+00
2.0	0.13534E+00	0.12158E+00	0.14864E+00	0.13684E+00	0.13533E+00
2.1	0.12246E+00	0.10942E+00	0.13513E+00	0.12068E+00	0.12245E+00
2.2	0.11080E+00	0.98477E+00	0.12285E+00	0.11244E+00	0.11080E+00
2.3	0.10026E+00	0.88629E+01	0.11168E+00	0.98386E+01	0.10026E+00
2.4	0.90718E+01	0.79766E+01	0.10153E+00	0.92491E+01	0.90715E+01
2.5	0.82085E+01	0.71790E+01	0.92297E+01	0.80104E+01	0.82082E+01
2.6	0.74274E+01	0.64611E+01	0.83906E+01	0.76188E+01	0.74271E+01
2.7	0.67206E+01	0.58149E+01	0.76278E+01	0.65106E+01	0.67203E+01
2.8	0.60810E+01	0.52335E+01	0.69344E+01	0.62872E+01	0.60808E+01
2.9	0.55023E+01	0.47101E+01	0.63040E+01	0.52793E+01	0.55021E+01
3.0	0.49787E+01	0.42391E+01	0.57309E+01	0.52004E+01	0.49785E+01

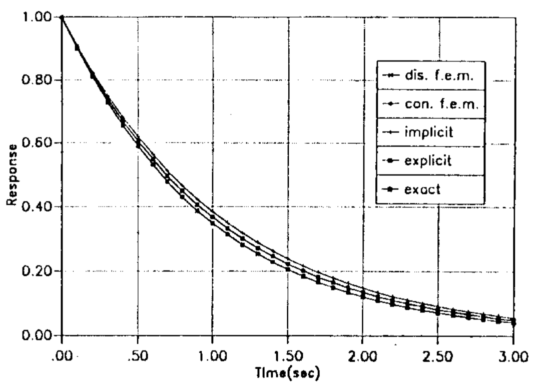

Figure 2. Comparison of numerical and exact solutions : $\Delta t=0.1$ sec

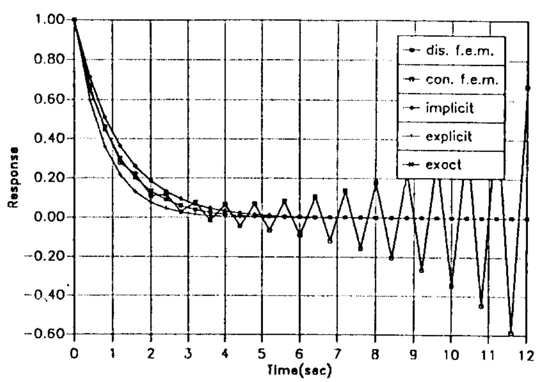

Figure 3. Comparison of numerical and exact solutions : $\Delta t=0.4$ sec

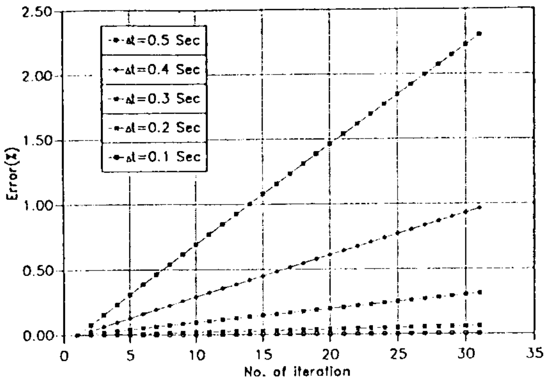

Figure 4. Error distribution with the discontinuous Galerkin method

ution. With $\Delta t = 0.4$ the results of continuous finite element method oscillate with growing oscillations. There is no plottable difference between the exact solution and the results of the discontinuous finite element method. Figure 4 illustrate the growth of the error in the discontinuous finite element method for the different time steps. Even with $\Delta t = 0.5$ second which is one half of the time constant of the exact solution these results are within 2.3% of the exact solution of $\exp(-16.5)$ at $t = 16.5$ seconds. This is amazing accuracy and is the reason that the time discontinuous finite element method was selected for this work.

4. Conclusions

Most of the early works on the discontinuous Galerkin method for ordinary differential equations have been mathematically oriented and proved. The aim of this study was to investigate the development of a time-discontinuous finite element formulation capable of solving a wide range of time-dependent problems. We have demonstrated the efficiency of using finite elements by allowing discontinuities with the respect to the time variables at each time step. We showed that the time-discontinuous Galerkin methods lead to a unconditional stable higher accurate ordinary differential equation solver. This is in contrast to the conditional stability of some time-continuous Galerkin methods. The time-discontinuous Galerkin method seems conducive to the establishment of rigorous convergence proof and error estimates.

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