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ON SPACES WITHOUT R^i – CONTINUA

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PREMINARY. *Let X be a metric continuum. Let $\mathcal{C}(X)$ be the hyperspace of all nonempty subcontinua of X with the Hausdorff metric H defined by $H(A,B) = \inf\{\epsilon > 0 : A \subset N(\epsilon, B)$ and $B \subset N(\epsilon, A)\}$, for $A, B \in \mathcal{C}(X)$,where $N(\epsilon, A)$ denotes the ϵ - neighborhood of A. For each $x \in X$, let $T(x) = \{A \in \mathcal{C}(X) : x \in A\}$. Then $T(x)$ is a closed subset of $\mathcal{C}(X)$. We call $T(x)$ the total fiber at x. An element $A \in T(x)$ is said to be admissible at x in X if, for each $\epsilon > 0$, there is a $\delta > 0$ such that each point y of the δ-neighborhood of x has an element $B \in T(y)$ such that $H(A, B) < \epsilon$. For each $x \in X$, we let $\mathcal{A}(x)$ be the collection of all admissible element at x in X. Then $\mathcal{A}(x)$ is closed in $\mathcal{C}(X)$ and $\{x\}$, $X \in \mathcal{A}(x)$. We call $\mathcal{A}(x)$ the admissible fiber at x. Let $M = \{x \in X : T(x) \neq \mathcal{A}(x)\}$ and call it the $\mathcal{M}$ -set of X. If the $\mathcal{M}$-set M of X is not empty then the components of M are nondegenerate. Let $\mu : \mathcal{C}(X) \to [0,1]$ be a Whitney map with $\mu(X) = 1$ [5]. We now define a metric continuum X to be T-admissible if, for each $(x,s) \in X \times [0,1]$ the following condition is true: for each $A \in \mathcal{A}(x) \cap \mu^{-1}(s)$ and $t \in [s,1]$, there is an element $B \in \mathcal{A}(x) \cap \mu^{-1}(t)$ such that $A \subset B$. For more about the notion of admissibility and $\mathcal{M}$ -set we refer [6]. The notion of R^i-continua is given in [2].*

PROPOSITION 1. *An element $A \in \mathcal{A}(x)$ if and only if for each ϵ-neighborhood U of A in X there is a δ -neighborhood V of x such that for each point $y \in V$ with the component C_y of U containing y, there is an element $B \in T(y)$ with $B \subset C_y$ such that $H(A, B) < \epsilon$.*

Proof. Suppose $A \in \mathcal{A}(x)$. Let $U = N(\epsilon, A)$. Then by the admissibility of A at x in X , there is a $\delta > 0$ suh that each point y in the δ -neighborhood V of x has an element $B \in T(y)$ such that $H(A, B) < \epsilon$.

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Let $\mathcal{C}_y$ be the component of U containing y. Then $B \cap \mathcal{C}_y \neq \emptyset$. If there is a point $z \in B\backslash U$, then $d(z, A) \geq \epsilon$ so that $H(A, B)$ would be greater or equal to ϵ. Therefore we conclude that B must be contained in U. Hence $B \subset \mathcal{C}_y$. The converse is obvious.

Let $\{\mathcal{C}_n\}$ be a sequence of subsets of a space. Denote the limit superior of the sequence by $L_s\mathcal{C}_n$, the limit inferior of it by $L_i\mathcal{C}_n$, and limit of the sequence by $L_t\mathcal{C}_n$. It is known [4] that : every sequence of sets has a convergent subsequence (the limit may be empty) ; for any subsequence $\{\mathcal{C}_{n_k}\}$ of $\{\mathcal{C}_n\}$, $L_i\mathcal{C}_n \subset L_i\mathcal{C}_{n_k} \subset L_s\mathcal{C}_{n_k} \subset L_s\mathcal{C}_n$; if each element $\mathcal{C}_n$ of $\{\mathcal{C}_n\}$ is connected and $L_i\mathcal{C}_n \neq \emptyset$ then $L_s\mathcal{C}_n$ is connected.

We say that a nonempty proper subcontinuum K of a metric continuum X is
-an R^1-continuum [2, Definition 1.1] if there exist an open set U containing K and two sequences $\{\mathcal{C}_n^1\}$, $\{\mathcal{C}_n^2\}$ of components of U such that $K = L_s\mathcal{C}_n^1 \cap L_s\mathcal{C}_n^2$;
-an R^2-continuum [2, Definition 1.2] if there exist an open set U containing K and two sequence $\{\mathcal{C}_n^1\}$, $\{\mathcal{C}_n^2\}$ of components of U such $K = L_t\mathcal{C}_n^1 \cap L_t\mathcal{C}_n^2$;
-an R^3-continuum [2, Definition 1.3] if there exist an open set U containing K and a sequence $\{\mathcal{C}_n\}$ of components of U such that $K = L_i\mathcal{C}_n$.

An order arc α in $\mathcal{C}(X)$ [5] is an arc in $\mathcal{C}(X)$ such that whenever $A, B \in \alpha$ either $A \subset B$ or $B \subset A$. An order arc α from $\{x\}$ to X in $\mathcal{C}(X)$ is said to be admissible at $x \in X$ if $\alpha \subset \mathcal{A}(x)$.

Thus if $\mathcal{A}(x)$ is path-connected, $\mathcal{A}(x)$ contains an admissible order arc from $\{x\}$ to X.

PROPOSION 2. *If X is a metric continuum such that $\mathcal{A}(x)$ contains an order arc from $\{x\}$ to X for each $x \in X$, then X is T -admissible.*

Proof. Let $A \in \mathcal{A}(x)$ and $t \in [\mu(A), 1]$. Then let $\alpha = \{A_s\}_{s\in[0,1]}$ be an order arc in $A(x)$ from $\{x\}$ to X. Let $\beta = \{A \cup A_s : A_s \in \alpha\}$. Then for each $s \in [0, 1]$, $A \cup A_s \in \mathcal{A}(x)$ by [6, Theorem 1.0.] so that $\beta \subset \mathcal{A}(x)$. It is easy to see that β is also an order arc from A to X.

Since $\mu(\beta) = [\mu(A), 1]$, there is an element $B \in \beta$ such that $\mu(B) = t$ and $A \subset B$. This proves that X is a T -admissible space.

PROPOSION 3. *If K is an R^i -continuum , $i = 1,2,3,$ then K is contained in the $\mathcal{M}$ -set of X.*

Proof. We prove only for R^1 -continuum. The proofs for the other R^i-continua are similar. Let U be an open set containing an R^1 - continuum K and let $\{\mathcal{C}_n^i\}$, $i = 1,2,$ be sequences of components of U such that $K = L_s\mathcal{C}_n^1 \cap L_s\mathcal{C}_n^2$, and let $\mathcal{C}_0$ be the component of U containing K. Let $\epsilon > 0$ such that the closure of $N(\epsilon, K)$ is contained in U. Let F be the closure of the component of $N(\epsilon, K)$ containing K. Then $F \subset \mathcal{C}_0 \subset U$ and $F \backslash K \neq \emptyset$. Let x be any point of K and let $\{x_n^i\}$, $i = 1,2,$ be sequence of points, $x_n^i \in \mathcal{C}_n^i$, such that as $n \to \infty$, $x_n^i \to x$, for each $i = 1,2$. If F is admissible at x in X , then by Proposition 1 there are two sequences $\{B_n^i\}$, $B_n^i \in T(x_n^i)$, $B_n^i \subset \mathcal{C}_n^i$, that $B_n^i \to F$ for each $i = 1,2$. This imply that $F \subset L_s\mathcal{C}_n^1 \cap L_s\mathcal{C}_n^2$ which contradicts the fact that $F \backslash K \neq \emptyset$. Therefore $F \notin \mathcal{A}(x)$. Hence x must belong to the $\mathcal{M}$ -set of X.

PROPOSION 4. *Let K be any one of R^i-continuum, $i = 1,2,3$. Let U be an open set containing K which has components which define K. If A is any subcontinuum of X contained in U such that A contains K properly, then $A \notin \mathcal{A}(x)$ for each $x \in K$.*

Proof. We prove for an R^2 -continuum. The proofs for other R^i continua are similar. Let $\{\mathcal{C}_n^1\}$ and $\{\mathcal{C}_n^2\}$ be two sequencs of components of U such that $K = L_t\mathcal{C}_n^1 \cap L_t\mathcal{C}_n^2$ and $x \in K$. Let $\{x_n\}$ be a sequence of points, $x_n \in \mathcal{C}_n^1$, which converges to x. Suppose $A \in \mathcal{A}(x)$ which contains K properly. Let $0 < \epsilon < H(A, X\backslash U)$. Then by the admissibility of A at x in X there is a sequence $\{B_n\}$, $B_n \in T(x_n)$, which converges to A. Let N be a positive integer such that $H(A, B_n) < \epsilon$ for $n \geq N$. Then for each $n \geq N$, $B_n \subset N(\epsilon, A)$ so that $B_n \subset \mathcal{C}_n^1$. This implies that $A \subset L_t\mathcal{C}_n^1$. Similarly there is a sequence $\{D_n\}$, $D_n \subset \mathcal{C}_n^2$, which converges to A. So that $A \subset L_t\mathcal{C}_n^2$. By combining these two we have $A \subset K$, Which is a contradition.

PROPOSITION 5. *Let X be a T-admissible continuum. Then X does not contain any R^i-continuum.*

Proof. Suppose X contains an R^3 continuum K. Let U be an open set containing K and let $\{C_n\}$ be a sequence of components of U such that $K = L_i C_n$. Let $x \in K$. Let $\mathcal{F}_x = \{B \in \mathcal{A}(x) : B \subset U\}$. Then $\mathcal{F}_x \neq \emptyset$. We claim that there is an element $A \in \mathcal{F}_x$ such that $A \setminus K \neq \emptyset$. We suppose to the contrary. Then each element of $\mathcal{F}_x$ must be contained in K. Since K is a subcontinuum of X. $F = \cup \mathcal{F}_x \in \mathcal{A}(x)$ by [6, Theorem 1.0] and $F \subset K$. Let $0 < \epsilon < H(F, X \setminus U)$. Then by [3, Theorem 2.2], there is a $\delta > 0$ such that if $F, B \in C(X)$ and $\mu(B) - \mu(F) < \delta$ and $F \subset N(\delta, B)$, then $H(F, B) < \epsilon$. So, by the T-admissibility of X, we choose an element $C \in \mathcal{A}(x)$ such that $F \subset C$ and $0 < \mu(C) - \mu(F) < \delta$. Then $H(F, C) < \epsilon$ implies that $C \subset N(\epsilon, F) \subset U$. Since $C \setminus F \neq 0$ and $C \notin \mathcal{F}_x$, this is a contradiction. So we must have an element $A \in \mathcal{F}_x$ such that $A \setminus K \neq \phi$. Now let A be such an element. Let $y \in A \setminus K$. And let $\{x_n\}$ be a sequence, $x_n \in C_n$, which converges to x. Since $A \in \mathcal{A}(x)$, there is a sequence $\{B_n\}, B_n \in T(x_n)$ which converges to A. Hence there is a positive integer N such that $H(A, B_n) < H(A, X \setminus U)$ and $B_n \subset N(H(A, X \setminus U), A)$ for each $n > N$. This imply that $A \subset L_i C_i$, which contradicts the fact that K is an R^3-continuum. This prove that X does not contain any R^3-continuum. Similar argument can be applied for the case when K is R^i-continuum, $i = 1, 2$.

COROLLARY 6 [1, COROLLARY 16]. *If the admissible fiber $\mathcal{A}(x)$ is path-connected for each $x \in X$, then X does not contain any R^i-continuum for $i = 1, 2, 3$.*

Proof. . Let $f : [0,1] \to \mathcal{A}(x)$ be a path such that $f(0) = \{x\}$ and $f(1) = X$. For each $t \in [0,1]$, let $A_t = \cup\{f(s) : 0 \leq s \leq t\}$. Then $A_t \in \mathcal{A}(x)$ for each t and the set $\beta = \{A_t : t \in [0,1]\}$ is an order arc. Hence the conclusion follows from Propositions 2 and 5.

REMARK. (a) Charatonik's Proposition 1 in [1] is wrong. He used it to prove his Corollary 16.
(b) The absence of R^i-continuum in the metric continuum X does not imply that X is T-admissible. (see the example in [1, Proposition 17].)

QUESTION 1. *Let X be a T-admissible space. Does the admissible fiber $\mathcal{A}(x)$ contain an order arc from $\{x\}$ to X ?*

QUESTION 2. *Does the hyperspace $C(X)$ of T-admissible space X contain any R^i-continuum ? (compare Question 21 in [1].)*

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