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CYCLIC SUBGROUP SEPARABILITY OF HNN EXTENSIONS

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1. Introduction

A group G is said to be *cyclic subgroup separable* (π_c) if, for each cyclic subgroup $\langle x\rangle$ of G, and for each element $g \in G\backslash\langle x\rangle$, there exists $N \lhd_f G$ such that $g \notin N\langle x\rangle$.

In [4], Baumslag and Tretkoff proved a residual finiteness criterion for HNN extensions (Theorem 1.2, below). This result has been used extensively in the study of the residual finiteness of HNN extensions. Note that every one-relator group can be embedded in a one-relator group whose relator has zero exponent sum on a generator, and the latter group can be considered as an HNN extension. Hence the properties of an HNN extension play an important role in the study of one-relator groups [3], [2]. In this paper we prove a criterion for HNN extensions to be π_c (Theorem 2.2). Moreover, we can prove that certain one-relator groups, known to be residually finite, are actually π_c.

It was known by Mostowski [10] that the word problem is solvable for finitely presented, residually finite groups. In the same way, the power problem is solvable for finitely presented π_c groups. Another application of subgroup separability with respect to special subgroups was mentioned by Thurston [12, Problem 15].

We shall adopt the following notations and terminology:

We use $N \lhd_f G$ to denote that N is a normal subgroup of finite index in G. "f.g." means "finitely generated". If $\overline{G}$ is a homomorphic image of G, then we use $\overline{x}$ to denote the image of $x \in G$ in $\overline{G}$. We denote by $\langle A, t\ ;\ t^{-1}ht = h\varphi,\ h \in H\rangle$ an HNN extension of a base group A, with stable letter t, and associated subgroups H and K, where $\varphi : H \to K$ is an isomorphism.

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Let H be a subgroup of a group G. Then G is said to be ***H-separable*** if, for each $x \in G\backslash H$, there exists $N \lhd_f G$ such that $x \notin NH$. If G is $\langle 1\rangle$-separable, then we say that G is ***residually finite*** ($\mathcal{RF}$). A group G is said to be ***subgroup separable*** if G is H-separable for all f.g. subgroups H of G.

For example, it is not difficult to see that a finite extension of a free group is subgroup separable, since a free group is subgroup separable [5]. Hence, we derive the following result:

THEOREM 1.1. [11] *Let $G = \langle A, t; t^{-1}ht = h\varphi,\ h \in H\rangle$ be an HNN extension and assume that A is a finite group. Then G is a finite extension of a free group, and so, in particular, G is subgroup separable.*

Next result is equivalent to Baumslag and Tretkoff's criterion [11].

THEOREM 1.2. [4] *Let $G = \langle A, t;\ t^{-1}ht = h\varphi,\ h \in H\rangle$ be an HNN extension. Let $\Delta = \{S \lhd_f A : (S\cap H)\varphi = S \cap K\}$. Assume that*
(a) $\cap_{S\in\Delta} HS = H$ and $\cap_{S\in\Delta} KS = K$,
(b) $\cap_{S\in\Delta} S = \langle 1\rangle$.
Then G is $\mathcal{RF}$.

Let G and Δ be as in Theorem 1.2. Then, for each $S \in \Delta$, we have a homomorphism

$$\phi_S : G \to \langle A/S, t_S;\ t_S^{-1}\overline{h}t_S = \overline{h}\overline{\varphi},\ \overline{h} \in \overline{H}\rangle, \tag{1}$$

where $\overline{A} = A/S$, $t\phi_S = t_S$ and $\overline{\varphi} : HS/S \to KS/S$ is an isomorphism induced by φ.

THEOREM 1.3. [11] *Let G and Δ be as in Theorem 1.2. Then G is $\mathcal{RF}$ if and only if $\cap_{S\in\Delta}$ Ker $\phi_S = \langle 1\rangle$.*

2. On HNN extensions

We recall some basic facts for HNN extensions, which may be found in [9] or [8].

REMARK 2.1. *Let $G = \langle A, t; t^{-1}ht = h\varphi,\ h \in H\rangle$ be an HNN extension.*

(1) *Each element $g \in G$ may be written in a reduced form $g = a_0t^{\epsilon_1}a_1t^{\epsilon_2}\cdots a_{n-1}t^{\epsilon_n}a_n$, where $a_i \in A$, $\epsilon_i = \pm 1$, and no subwords $t^{-1}ht$ $(h \in H)$ or tkt^{-1} $(k \in K)$ occur.*

(2) *Let $g = a_0t^{\epsilon_1}\cdots t^{\epsilon_n}a_n$ be a reduced form as above. Then we define the length of g, written $\|g\|$, as the number n of occurrences of t and t^{-1} in g.*

(3) *An element $g = a_0t^{\epsilon_1}\cdots a_{n-1}t^{\epsilon_n}$ is said to be cyclically reduced if all cyclic permutations, $a_{i-1}t^{\epsilon_i}a_i\cdots a_{n-1}t^{\epsilon_n}\cdot a_0t^{\epsilon_1}a_1\cdots a_{i-2}t^{\epsilon_{i-1}}$, of g are reduced. Clearly, every element of G is conjugate to a cyclically reduced form.*

Now we are ready to prove our main theorem of this paper.

THEOREM 2.2. *Let $G = \langle A, t; t^{-1}ht = h\varphi,\ h \in H\rangle$ be an HNN extension. Let $\Delta = \{P \lhd_f A : (P \cap H)\varphi = P \cap K\}$. Assume that*

(a) $\cap_{P\in\Delta}HP = H$ and $\cap_{P\in\Delta}KP = K$,

(b) $\cap_{P\in\Delta}P\langle x\rangle = \langle x\rangle$ for all $x \in A$.

Then G is π_c.

Proof. Let g, x be reduced forms in G such that $g \notin \langle x\rangle$. Since every element in G is conjugate to a cyclically reduced form, we may assume that x is cyclically reduced. Moreover, since G is $\mathcal{RF}$ by Theorem 1.2, we may assume $x \neq 1$.

Case 1. Suppose $g \notin \langle x\rangle$ is implied by the syllable length of g and x; that is,

subcase 1 $\|x\| = 0$ and $\|g\| \geq 1$,

subcase 2 $\|x\| \geq 1$ and $\|g\| = 0$,

subcase 3 $\|x\| \geq 1$, $\|g\| \neq 0$ and $\|x\|$ does not divide $\|g\|$.

For these subcases, we can find $S \in \Delta$ such that $\overline{g}$ is reduced, $\|\overline{g}\| = \|g\|$, $\overline{g} \neq 1$, and that $\overline{x}$ is cyclically reduced, $\|\overline{x}\| = \|x\|$, $\overline{x} \neq 1$, where $\overline{G} = G\phi_S = \langle A/S, t_S;\ t_S^{-1}\overline{h}t_S = \overline{h\varphi},\ \overline{h} \in \overline{H}\rangle$ is as in (1, p.2). It follows that $\overline{g} \notin \langle\overline{x}\rangle$. Since $\overline{G}$ is π_c by Theorem 1.1, there exists $\overline{N} \lhd_f \overline{G}$ such that $\overline{g} \notin \overline{N}\langle\overline{x}\rangle$. Let N be the preimage of $\overline{N}$ in G. Then $g \notin N\langle x\rangle$ and $N \lhd_f G$, as required.

Case 2. $\|x\| = 0 = \|g\|$. Then, by assumption (b), there exists $S \in \Delta$

such that $g \notin S\langle x\rangle$. Considering $\overline{G} = G\phi_S$ as before, we have $\overline{g} \notin \langle\overline{x}\rangle$, and hence, we can find $N \lhd_f G$ such that $g \notin N\langle x\rangle$, as required.

Case 3. $\|x\| \geq 1$, $\|g\| \neq 0$ *and* $\|x\|$ *divides* $\|g\|$. Since x is cyclically reduced, we may assume that $x = a_0 t^{\delta_1} a_1 t^{\delta_2} \cdots a_{n-1} t^{\delta_n}$, where $a_j \in A$, $n \geq 1$, and $\delta_{j+1} = \pm 1$. Let $\|g\| = m = ns$ and let $g = b_0 t^{\epsilon_1} b_1 t^{\epsilon_2} \cdots b_{m-1} t^{\epsilon_m} b_m$ be reduced, where $b_i \in A$ and $\epsilon_i = \pm 1$. By (a), we can find $S_1 \in \Delta$ such that $a_i \notin S_1 H$ if $a_i \notin H$, or $a_i \notin S_1 K$ if $a_i \notin K$, for each i. Similarly, we can find $S_2 \in \Delta$ such that $b_j \notin S_2 H$ if $b_j \notin H$, or $b_j \notin S_2 K$ if $b_j \notin K$, for each j. Now, since $g^{-1}x^s \neq 1 \neq gx^s$ and since G is $\mathcal{RF}$ (by Theorem 1.2), there exists $M \lhd_f G$ such that $g^{-1}x^s \notin M$ and $gx^s \notin M$. Then $M \cap A \in \Delta$ and $P = S_1 \cap S_2 \cap (M \cap A) \in \Delta$. Since $P \subset S_1 \cap S_2$, $\overline{g}$ is reduced and $\overline{x}$ is cyclically reduced, where $\overline{G} = G\phi_P$. Moreover, we have $\|\overline{g}\| = \|g\| = m = ns = \|x^s\| = \|\overline{x}^s\|$ and $\overline{g} \neq \overline{x}^{\pm s}$, where $\overline{G} = G\phi_P$. It follows that $\overline{g} \notin \langle\overline{x}\rangle$. Then, as in Case 1, we can find $N \lhd_f G$ such that $g \notin N\langle x\rangle$. This completes the proof.

COROLLARY 2.3. *Suppose that H and K are finite and that A is π_c. Then the HNN extension $G = \langle A, t; t^{-1}ht = h\varphi,\ h \in H\rangle$ is π_c.*

Proof. To apply Theorem 2.2, we prove (a) and (b) in the theorem. To prove (a), let $a \in A\backslash H$. Since A is $\mathcal{RF}$ and H, K are finite, there exists $P \lhd_f A$ such that $Ha \cap P = \emptyset$ and $P \cap H = 1 = P \cap K$. Then $P \in \Delta$ and $a \notin HP$. This proves that $\cap_{P\in\Delta} HP = H$. Similarly, $\cap_{P\in\Delta} KP = K$.

To prove (b), let $a, x \in A$ be such that $a \notin \langle x\rangle$. Since H and K are finite and A is π_c, there exists $P \lhd_f A$ such that $a \notin P\langle x\rangle$ and $P \cap H = 1 = P \cap K$, hence $P \in \Delta$. This proves (b). Therefore G is π_c by Theorem 2.2.

COROLLARY 2.4. *If $\varphi : A \to A$ is an automorphism and A is f.g., π_c, and H-separable, then $G = \langle A, t; t^{-1}ht = h\varphi,\ h \in H\rangle$ is π_c.*

Proof. To apply Theorem 2.2, we check conditions (a) and (b) in the theorem.

To prove (a), let $a \in A\backslash H$. Since A is H-separable, there exists $N \lhd_f A$ such that $a \notin HN$. Now A is f.g. It follows that there exists a characteristic subgroup P of A such that $P \subset N$ and $P \lhd_f A$. Hence $a \notin HP$. Since P is characteristic in A, we have $P\varphi = P$. Thus $(P \cap H)\varphi = P \cap K$. It follows that $P \in \Delta$ and $a \notin HP$, proving (a).

To prove (b), let $a, x \in A$ be such that $a \notin \langle x \rangle$. Since A is π_c, there exists $N \lhd_f A$ such that $a \notin N\langle x \rangle$. As before we can find a characteristic subgroup P of A such that $P \subset N$ and $P \lhd_f A$. Then $P \in \Delta$ and $a \notin P\langle x \rangle$ as before. This proves (b). Hence G is π_c by Theorem 2.2.

COROLLARY 2.5. *Let $\varphi : A \to A$ be an inner automorphism and suppose that A is π_c and H-separable. Then $\langle A, t; t^{-1}ht = h\varphi,\ h \in H\rangle$ is π_c.*

Proof. We note that $(N \cap H)\varphi = N \cap K$ for each $N \lhd_f A$. Thus, $N \in \Delta$, if $N \lhd_f A$. Now, the proof is similar to that of the above corollary.

In the above corollaries, H-separability is necessary in the following sense (see [11, Theorem 1]):

COROLLARY 2.6. *Let A be π_c. Then the HNN extension $G = \langle A, t; t^{-1}ht = h,\ h \in H\rangle$ is π_c if, and only if, A is H-separable.*

For the rest of this section, we recall the homomorphism ϕ_S given by (1, p.2). We extend Theorem 1.3.

THEOREM 2.7. *Let G and Δ be as in Theorem 2.2. For a given f.g. subgroup L of G, G is L-separable if, and only if, $\cap_{S\in\Delta}(\text{Ker } \phi_S)L = L$.*

Proof. ($\Longleftarrow$) Let $g \notin L$, where $g \in G$. Then, by assumption, there exists $P \in \Delta$ such that $g \notin (\text{Ker } \phi_P)L$. Thus $g\phi_P \notin L\phi_P$ and $L\phi_P$ is f.g. Since $G\phi_P$ is subgroup separable by Theorem 1.1, there exists $\overline{N} \lhd_f G\phi_P$ such that $g\phi_P \notin \overline{N}(L\phi_P)$. Let N be the preimage of $\overline{N}$ in G. Then $N \lhd_f G$ and $g \notin NL$, as required.

($\Longrightarrow$) Assume that G is L-separable and that $g \notin L$. Then there exists $N \lhd_f G$ such that $g \notin NL$. Let $P = N \cap A$. Then $P \lhd_f A$ and $(P \cap H)\varphi = P \cap K$, since $N \lhd_f G$. Thus, $\text{Ker } \phi_P \subset N$, and hence, $g \notin (\text{Ker } \phi_P)L$. This proves that $\cap_{S\in\Delta}(\text{Ker } \phi_S)L \subset L$. Hence, $\cap_{S\in\Delta}(\text{Ker } \phi_S)L = L$.

COROLLARY 2.8. *Let G and Δ be as in Theorem 2.2. Then G is π_c if, and only if, $\cap_{S\in\Delta}$ $(\text{Ker } \phi_S)\langle g\rangle = \langle g\rangle$ for all $g \in G$.*

COROLLARY 2.9. *Let G and Δ be as in Theorem 2.2 Assume that* $\cap_{S\in\Delta} HS = H$ *and* $\cap_{S\in\Delta} KS = K$. *Then G is π_c if, and only if,* $\cap_{S\in\Delta}$ (Ker $\phi_S)\langle x\rangle = \langle x\rangle$ *for all $x \in A$.*

Proof. This follows directly from Theorem 2.2 and Corollary 2.8.

Finally we note that Larsen [7] showed that HNN extensions of f.g. free groups with cyclic associated subgroups have solvable power problem. A finitely presented π_c group has solvable power problem. But, as in [1], the group $\langle a, b; b^{-1}ab = a^2\rangle$ is not π_c.

3. One-relator groups

In [3], [2], Allenby and Tang proved that the one-relator groups in this section are residually finite. Using our criterion, we prove that these groups are π_c. First we note the following results for the generalized free product of groups.

THEOREM 3.1. [6] *Let $G = A *_H B$ be a generalized free product of the groups A and B, amalgamating the subgroup H, and let* $\Lambda = \{(P,Q) : P \lhd_f A,\ Q \lhd_f B$ and $P \cap H = Q \cap H\}$. *Assume that*
(1) $\cap_{(P,Q)\in\Lambda} PH = H$ *and* $\cap_{(P,Q)\in\Lambda} QH = H$,
(2) $\cap_{(P,Q)\in\Lambda} P\langle x\rangle = \langle x\rangle$ *and* $\cap_{(P,Q)\in\Lambda} Q\langle y\rangle = \langle y\rangle$ *for all $x \in A$, $y \in B$.*
Then G is π_c.

A group G is said to be $\langle x\rangle$-*potent* if, for each positive integer n, there exists $N \lhd_f G$ such that Nx has order exactly n in G/N. The following result is analogous to Theorem A-T in [2].

COROLLARY 3.2. *Let A and B be π_c and let A be $\langle c^f\rangle$-potent, for some integer f. Then the generalized free product $A *_{\langle c\rangle} B$ of A and B, amalgamating $\langle c\rangle$, is π_c.*

Proof. To apply the above theorem, we prove the following facts:
1. For each $N \lhd_f A$, there exists $(P,Q) \in \Lambda$ such that $P \subset N$.
Let $N \cap \langle c\rangle = \langle c^k\rangle$. Since $c^i \notin \langle c^{fk}\rangle$, for all $1 \le i < fk$, there exists $N_1 \lhd_f A$ such that $c^i \notin N_1\langle c^{fk}\rangle$, for all i. Thus $N_1 \cap \langle c\rangle = \langle c^{fkt}\rangle$, for some t. Similarly, there exists $M_1 \lhd_f B$ such that $c^j \notin M_1\langle c^{fkt}\rangle$, for all $1 \le j < fkt$. Thus $M_1 \cap \langle c\rangle = \langle c^{fktm}\rangle$, for some m. Since A is

$\langle c^f \rangle$-potent, there exists $N_2 \lhd_f A$ such that $N_2 \cap \langle c^f \rangle = \langle c^{fktm} \rangle$. Let $P = N \cap N_1 \cap N_2$ and $Q = M_1$. Then $P \cap \langle c \rangle = \langle c^{fktm} \rangle = Q \cap \langle c \rangle$, and hence, $(P, Q) \in \Lambda$ and $P \subset N$.

2. For each $M \lhd_f B$, there exists $(P, Q) \in \Lambda$ such that $Q \subset M$.

Let $M \cap \langle c \rangle = \langle c^k \rangle$. Let N_1, N_2, and M_1 be as above. Then $P = N_1 \cap N_2$ and $Q = M \cap M_1$ satisfy our requirement.

Now it is not difficult to apply Theorem 3.1

The residual finiteness of the groups G in the next two results was known to Allenby and Tang [3]. Using Theorem 2.2, we prove that these groups are actually π_c.

THEOREM 3.3. *Let $G = \langle a, b; (r(a,b))^t \rangle$, where $t \geq 1$ and $r(a,b)$ is a cyclically reduced word on a and b, with b exponent sum equal to zero, that is not a proper power. Regarding G as an HNN extension by $\langle b \rangle$ of the base group $A = \langle a_L, a_{L+1}, \ldots, a_M; (\overline{r}(a_i))^t \rangle$, where $a_i = b^{-i}ab^i$, if we find that both a_L and a_M occur only once in $\overline{r}(a_i)$, (where L and M are respectively the smallest and largest indices occurring in $\overline{r}(a_i)$), then G is π_c.*

Proof. First we note that if $t = 1$, then G is a cyclic extension of a f.g. free group which is π_c by [1]. As explained in [3], G has associated subgroups $H = \langle a_L, a_{L+1}, \ldots, a_{M-1} \rangle$ and $K = \langle a_{L+1}, \ldots, a_M \rangle$, where $\varphi : H \to K$ is the isomorphism defined by $a_i\varphi = a_{i+1}$. In [3], we find the following facts, for $t \geq 2$:

1. A has the property (a) in Theorem 2.2.
2. If P is a characteristic subgroup of A with finite index, then $(P \cap H)\varphi = P \cap K$.
3. A is subgroup separable.

Thus we need only show (b) in Theorem 2.2. For this, let $g, x \in A$ be such that $g \notin \langle x \rangle$. Since A is π_c, there exists $N \lhd_f A$ such that $g \notin N\langle x \rangle$. Note that A is f.g. Hence, we can find a characteristic subgroup P of A with finite index in A such that $P \subset N$. This proves (b) by 2 above. Therefore G is π_c.

COROLLARY 3.4. *The group $G = \langle a, b; [a, b^{k_1}, \ldots, b^{k_n}]^t \rangle$ is π_c for $t \geq 1$.*

Proof. As explained in [3, Lemma 4.2], we note that $[a, b^{k_1}, \ldots, b^{k_n}]$ can be expressed as a product $a_{t_1}^{-1} a_{t_2} a_{t_3}^{-1} \cdots a_{t_{2^n}}$, where $a_i = b^{-i} a b^i$, and the 2^n suffices t_i are precisely the 2^n partial sums of $\{k_1, \ldots, k_n\}$, each appearing once. Hence, the corollary follows from Theorem 3.3.

Because of Corollary 3.2, and the similarity between Theorem 1.2 and Theorem 2.2, with only minor change of the proofs in [3] and [2], we can prove that the groups in the following theorems are still π_c. We denote by $u(b_i)$ and $v(b_i)$, words on the letters b_i, for the following theorems.

THEOREM 3.5. *The group* $G = \langle a, b_1, \ldots, b_k;\ (a^{-l} u(b_i) a^l v(b_i))^t \rangle$ *is* π_c *for* $t \geq 2$.

We note that the result for $l = 1$ in the above theorem was claimed by Shirvani (see footnote in [2]).

THEOREM 3.6. *The group* $G = \langle c_1, c_2, \ldots, c_m, d_1, d_2, \ldots, d_n;\ [u(c_i),\ v(d_j)]^s \rangle$ *is* π_c *for* $s \geq 1$.

THEOREM 3.7. *The group* $G = \langle c_1, c_2, \ldots, c_m, d_1, d_2, \ldots, d_n;\ (u(c_i)^\alpha v(d_j)^\beta)^s \rangle$ *is* π_c *for* $s \geq 1$ *and* $\alpha\beta \neq 0$.

THEOREM 3.8. *The group* $G = \langle c_1, c_2, \ldots, c_m, d_1, d_2, \ldots, d_n;\ (u(c_i)^\alpha v(d_j)^\beta u(c_i)^\gamma v(d_j)^\delta)^s \rangle$ *is* π_c *for* $s \geq 1$, *if* $\langle u, v\ ;\ (u^\alpha v^\beta u^\gamma v^\delta)^s \rangle$ *is* π_c.

A word w is said to be ***positive***, if only non-negative powers of the generators of the group occur in w.

THEOREM 3.9. *Let* $G = \langle g, h, \ldots, k;\ (uv^{-1})^s \rangle$, *where* u *and* v *are positive words on the generators* $g, h, \ldots, k$ *and where each generator appears in* uv^{-1} *with zero exponent sum. Then, for* $s \geq 1$, G *is* π_c.

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