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ON THE SOLVABILITY OF THE NONLINEAR FUNCTIONAL EQUATIONS IN BANACH SPACES

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1. Introduction

Let E be a real Banach space and let T be a (generally multivalued) operator in E. We consider the nonlinear functional equation:

$$\text{(E)} \qquad f \in Tu$$

The mapping theorems for the equation (E) with accretive operator T in Banach spaces have been studied by Kartsatos [8], Kirk and Schöneberg [10], and Morales [11]. The case of A-proper operator T has been given by Webb [15].

On the other hand, for $T = A + B$ with A and B two monotone operators in Hilbert space H, the equation (E) has been studied by Brézis and Haraux [3], Brézis and Nirenberg [4], and Gupta and Hess [7]. In the case of accretive eperator in Banach spaces with $T = A + B$, the equation (E) has been considered by Calvert and Gupta [5], Kartsatos [9], Reich [12] and Torrejón [13].

The purpose of this paper is to study the solvability of the equation (E). In Section 2, we give preliminary definitions. In Section 3, we prove related two results (Theorem 1 and Corollary 1) concerning the closedness property of accretive operators in the class of spaces whose nonempty bounded closed convex subsets have the fixed point property for nonexpansive self-mapping. Using Theorem 1, we derive a result (Theorem 2) on the range of accretive operators in $(\pi)_1$ spaces with a view to establishing a new result, which improves a result of Kartsatos

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[8] and Webb [15]. Further, we give an interesting consequence (Corollary 3) of Theorem 2. In Section 4, we apply Corollary 1 to obtain two results (Theorem 3 and 4) for the range of sums of two accretive operators, which generalize two results of Reich [12].

2. Preliminaries

Let E be a real Banach space and let E^* be its dual. The duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{j \in E^* : (x,j) = \|x\|^2 = \|j\|^2\},$$

and for each x and y in E we let

$$[y,x]_+ = \sup\{(y,j) : j \in J(x)\}$$

and

$$[y,x]_- = \inf\{(y,j) : j \in J(x)\}.$$

Since $J(x)$ is a weak-star compact subset of E^*, these values are actually attained. Furthermore, the following relationships hold:

(i) $[x+y,z]_\pm \le [x,z]_\pm + [y,z]_\pm$
(ii) $|[x,y]_\pm| \le \|x\|\|y\|$
(iii) $[x+\alpha y,y]_\pm = [x,y]_\pm + \alpha\|y\|^2$ for all $\alpha \in \mathbb{R}$
(iv) $[\alpha x,\beta y]_\pm = \alpha\beta[x,y]_\pm$ for all $\alpha,\beta \in \mathbb{R}$ with $\alpha\cdot\beta \ge 0$.

Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then a Banach space is said to be smooth provided the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. In this case, the norm of E is said to be Gâteaux differentiable. The space E is said to be uniformly smooth (or equivalently, E^* is uniformly convex) if this limit is attained uniformly for $(x,y) \in U \times U$. In this case, the space E is said to have uniformly

Fréchet differentiable norm. It is well known that E is smooth if and only if the duality mapping J is single valued.

Let I denote the identity operator. Recall that an operator $A \subset E \times E$ with domain $D(A)$ and range $R(A)$ is said to be accretive if $\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\|$ for all $[x_i, y_i] \in A, i = 1, 2$, and $\lambda > 0$. A is accretive if and only if for any $x_i \in D(A)$ and $y_i \in Ax_i, i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $(y_1 - y_2, j) \geq 0$ (or equivalently if and only if for any $x_i \in D(A)$ and $y_i \in Ax_i, i = 1, 2, [y_1 - y_2, x_1 - x_2]_+ \geq 0$). An accretive operator A is said to m-accretive if $R(I + \lambda A) = E$ for all $\lambda > 0$. For an m-accretive operator A, the resolvent $J_\lambda^A = (I + \lambda A)^{-1}, \lambda > 0$, is a single valued nonexpansive mapping which is defined on all of E.

Let C be a nonempty closed convex subset of E. Then C is said to have the fixed point property for nonexpansive self-mappings if for every nonexpansive mapping $T : C \to C$, there is a point $p \in C$ such that $T(p) = p$. It is known that every bounded closed convex subset of a uniformly smooth Banach space has the fixed point property for nonexpansive self-mappings(cf. [6, P. 45]).

The closure, interior and boundary of $D \subset E$ will be denoted by $\mathrm{cl}(D), \mathrm{int}(D)$ and ∂D, respectively. We will use $B_r(x)$ to denote the open ball centered at $x \in E$ with radius $r > 0$.

3. The mapping theorem for accretive operators

We start proving a result concerning the closedness property of accretive operators.

THEOREM 1. *Let E be a Banach space for which nonempty bounded closed convex subset has the fixed point property for nonexpansive self-mappings, and $f \in E$. If $A \subset E \times E$ is m-accretive, $y_n \in Ax_n, \{x_n\}$ is bounded and*

$$\lambda_n x_n + y_n = f + \lambda_n g \tag{1}$$

for $\lambda_n \to 0$ as $n \to \infty$ and some $g \in E$, then there exists $u \in D(A)$ such that $f \in Au$, that is, $f \in R(A)$.

Proof. Suppose that $y_n \in Ax_n$ such that $\{x_n\}$ is bounded and satisfies (1). Let $K = \limsup_{n\to\infty} \|x_n\|$ and set

$$M = \{x \in E : \limsup_{n\to\infty} \|x_n - x\| \le K\}.$$

Then M is a nonempty bounded closed convex subset of E. Now let $C = A - f$. Then $J_1^C x = J_1^A(x + f)$ for all $x \in E$. Furthermore, M is invariant under J_1^C. In fact, since $y_n \in Ax_n$ or $x_n + y_n \in (I + A)x_n$, we have $x_n = J_1^A(x_n + y_n)$. So if $x \in M$, then by (1),

$$\begin{aligned}\|x_n - J_1^C x\| &= \|J_1^A(x_n + y_n) - J_1^A(x + f)\| \\ &\le \|(x_n + y_n) - (x + f)\| \\ &= \|x_n - x\| + |\lambda_n|\|x_n - g\|\end{aligned}$$

and hence

$$\limsup_{n\to\infty} \|x_n - J_1^C x\| \le \limsup_{n\to\infty} \|x_n - x\| \le K$$

because $\lim_{n\to\infty} |\lambda_n|\|x_n - g\| = 0$. This implies that $J_1^C x \in M$, that is, $J_1^C M \subset M$. Thus by the nonexpansiveness of J_1^C and hypothesis, there $u \in M$ such that $J_1^C u = u$ or $J_1^A(u + f) = u$. This give $u + f \in (I + A)u$, that is, $f \in Au$.

Theorem 1 is an improvement of [16, Theorem 2] for multivalued operators although it does require the existence of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to u. The following result is also an improvement of [16, Theorem 1](cf. [12, Lemma 1.1]).

Corollary 1. *Let E be a Banach space for which each nonempty bounded closed convex subset has the fixed point property for nonexpansive self-mappings, and $f \in E$. If $A \subset E \times E$ is m-accretive, $y_n \in Ax_n, \{x_n\}$ is bounded and $y_n \to f$ as $n \to \infty$, then there exists $u \in D(A)$ such that $f \in Au$, that is, $f \in R(A)$.*

Proof. The conclusion of this corollary follows from the proof of Theorem 1.

Using Theorem 1, we prove a theorem on the range of accretive operators in $(\pi)_1$ spaces, which is analogous to Webb's result [15, Theorem 4]. For this purpose, we give the following definitions.

A Banach space E is called a $(\pi)_1$ space if there exists a sequence of linear projections $\{P_n\}$, each of norm one, with finite dimensional range $E_n = P_n E$ satisfying $E_n \subset E_{n+1}$ and such that $P_n x \to x$ as $n \to \infty$ for each x in E. A mapping $T : E \to E$ is said to be A-proper if $P_n T : E_n \to E_n$ is continuous for each n and whenever $\{x_{n_j}\}$ is a bounded sequence with $x_{n_j} \in E_{n_j}$ and $P_{n_j} T x_{n_j} \to f \in E$ as $j \to \infty$, then there exists a subsequence $\{x_{n_{j(k)}}\}$, say, which converges to a point x satisfying $Tx = f$. A mapping $T : E \to E$ is called demicontinuous if $x_n \to x$ implies that Tx_n converges weakly to Tx. An operator $A \subset E \times E$ is strongly accretive if $A - \lambda I$ is accretive for some $\lambda > 0$.

In [14, 15], Webb proved the following results.

LEMMA 1([14]). *Let E be a $(\pi)_1$ space with uniformly convex dual E^* and let $T : E \to E$ be demicontinuous and strongly accretive. Then T is A-proper and $R(T) = E$.*

LEMMA 2([15]). *Let G be an open bounded subset of a $(\pi)_1$ space E and let $T : \mathrm{cl}(G) \to E$ be A-proper (with obvious modifications of the definition). Write G_n for $G \cap E_n$, an open subset of E_n and let ∂G_n denote its boundary in E_n. Suppose there exist $w \in G$ and a sequence $n_j \to \infty$ such that, for all $x \in \partial G_{n_j}$ and all $t \in [0, 1]$,*

$$tP_{n_j} Tx + (1 - t)x \neq P_{n_j} w.$$

Then $T(G)$ contains every pathwise connected subset of E that contains w and does not intersect $T(\partial G)$.

THEOREM 2. *Let E be a $(\pi)_1$ space with uniformly convex dual E^* and let $T : E \to E$ be demicontinuous and accretive. Suppose that G is an open bounded neighborhood of $x_0 \in E$ such that for some $r > 0$*

$$\|Tx_0\| < r \leq \|Tx\| \quad \text{for all } x \in \partial G. \tag{2}$$

Then $\mathrm{cl}(B_r(0)) \subset R(T)$.

Proof. Without loss of generality, we may assume that $x_0 = 0$ in (2). In fact, we may replace T by T' given by $T'x = T(x + x_0)$ for all $x \in E$. Now we follow the argument of Webb in [15]. First suppose that T is strongly accretive with constant c. Then it follows from Lemma 1 that $\mu I + T$ is A-proper for all $\mu \geq 0$. We claim that for all n sufficiently large, and $x \in \partial G \cap E_n$ and $t \in [0, 1]$,

$$tP_nTx + (1-t)x \neq 0.$$

Indeed, otherwise there would be sequence $\{t_j\}$, $0 \leq t_j \leq 1$ and $\{x_j\} \in \partial G \cap E_j$ and such that

$$t_jP_jTx_j + (1-t_j)x_j = 0. \tag{3}$$

Clearly we cannot have $t_j = 0$ for any such j. For this sequence $\{x_j\}$, we have, using the fact that $P_j^*J(x) = J(x)$ for $x \in E_j$ (cf. [2]) and T is strongly accretive, we have

$$(P_jTx_j - P_jTz, J(x_j - z)) \geq c\|x_j - z\|^2 \quad \text{for all} \quad z \in E_j.$$

and so $\|x_j - z\| \leq \|x_j + \alpha P_jTx_j - (z + \alpha P_jTz)\|$ for all $\alpha > 0$. From this fact and (3), we have (the case $t_j = 1$ is trivial)

$$\|P_jTx_j\| \leq \|P_jT(0)\| \leq \|T(0)\|.$$

That is, $\{P_jTx_j\}$ is uniformly bounded. Passing to a subsequence, we can suppose that $t_j \to t_0$. The above fact shows that $t_0 \neq 0$. For $0 < t_0 \leq 1$, since $((1-t_0)/t_0)I + T$ is A-proper, there exists a convergent subsequence of $\{x_j\}$ whose limit $x \in \partial G$ satisfies $(1-t_0)x + t_0Tx = 0$. This implies that $\|Tx\| \leq \|T(0)\| < r$ which contradicts to (2). Since $B_r(0)$ is pathwise connected and never intersects $T(\partial G)$, Lemma 2 implies that $T(G) \supset B_r(0)$, that is, $B_r(0) \subset R(T)$. In general case, let $\lambda_n = \frac{1}{n}$. Then $T_n = \lambda_n I + T$ is strongly accretive. Moreover, for $x \in \partial G$ and $s = \sup\{\|x\| : x \in \partial G\}$,

$$\|T_n(0)\| < r - \lambda_n s \leq \|T_n(x)\|$$

for n sufficiently large. Thus $T_n(G) \supset B_{r-\lambda_n s}(0)$ for all sufficiently large n. Thus, given $f \in \mathrm{cl}(B_r(0))$, there exists $x_n \in G$ such that

$$\lambda_n x_n + T x_n = (1 - \frac{2s}{r}\lambda_n) f.$$

Therefore, by Theorem 1, there exists $x \in E$ such that $f = Tx$, that is, $f \in R(T)$.

Theorem 2 is an improvement of [8, Theorem 2] although those results do not require E to be a $(\pi)_1$ space. In particular, we wish to point out that E is not assumed to be uniformly convex. As a direct consequence, we also have the following result, which is partially an improvement of [15, Theorem 4].

COROLLARY 2. *Let E be a $(\pi)_1$ space with uniformly convex dual E^* and let $T : E \to E$ be demicontinuous and accretive. Suppose that there are positive constants r, s such that*

$$\|T(0)\| < r \leq \inf_{\|x\|=s} \|Tx\|.$$

Then $\mathrm{cl}(B_r(0)) \subset R(T)$.

We conclude this section with the following new result.

COROLLARY 3. *Let E be a $(\pi)_1$ space with uniformly convex dual E^* and let $T : E \to E$ be demicontinuous and accretive. Suppose that for some $\delta > 0$ the set*

$$F = \{x \in E : \|Tx\| < \delta\}$$

is nonempty and bounded. Then $\mathrm{cl}(B_\delta(0)) \subset R(T)$.

Proof. Since F is nonempty and bounded, there exist $x_0 \in E$ and an open ball $B_\mu(x_0)$ for some $\mu > 0$ such that $F \subset B_\mu(x_0)$. Therefore

$$\|Tx_0\| < \delta \leq \|Tx\| \text{ for all } x \in \partial B_\mu(x_0).$$

Then Theorem 2 implies that $\mathrm{cl}(B_\delta(0)) \subset R(T)$.

4. The range of sums of accretive operators

In this section, we obtain two results for the range of sums of two accretive operators. Applying Corollary 1, we prepare the following lemma in spirit of [3, Lemma 1](cf. [12]).

LEMMA 3. *Let E be a Banach space, $A \subset E \times E$ an m-accretive operator and F a subset of E. If there is $a \in E$ such that for all $f \in F$*

$$\sup_{[z,w]\in A} [w - f, a - z]_+ < \infty, \tag{4}$$

then $f \in \mathrm{cl}(R(A))$. If we assume further that each nonempty bounded closed convex subset of E has the fixed point property for nonexpansive self-mappings, then $\mathrm{int}(F) \subset R(A)$.

Proof. Let $f \in F$. For each $r > 0$, let $u_r \in E$ be such that $f \in Au_r + ru_r$. By (4), there exist $a \in E$ and $K \in \mathrm{R}$ such that

$$[w - f, a - z]_+ \le K$$

for all $[z, w] \in A$. Taking $z = u_r, w = f - ru_r$, we have $[-ru_r, a - u_r]_+ \le K$. This give that

$$\begin{aligned} r\|u_r - a\|^2 &\le K + r[a, a - u_r]_- \\ &\le K + r\|a\|\|a - u_r\| \\ &\le K + \frac{1}{2}r(\|a\|^2 + \|u_r - a\|^2). \end{aligned}$$

Consequently, $\sqrt{r}\|u_r\|$ is bounded as $r \to 0$ and hence $\{ru_r\} \to 0$, as $r \to 0$, proving that $f \in \mathrm{cl}(R(A))$.

Next, suppose that f belongs to $\mathrm{int}(F)$. For each $r > 0$, let $u_r \in E$ be such that $f \in Au_r + ru_r$. Now let $\rho > 0$ such that $\|h\| \le \rho$ implies $f + h \in F$. Apply (4) to $f + h$ and to $[u_r, f - ru_r] \in A$. Then there exist $a \in E$ and $K(h)$ such that

$$[w - f - h, a - u_r]_+ = [-ru_r - h, a - u_r]_+ \le K(h).$$

This gives that

$$\begin{aligned}[h, u_r - a]_+ &\le K(h) + r[u_r, a - u_r]_- \\ &= K(h) - r\|a - u_r\|^2 + r[a, a - u_r]_- \\ &\le K(h) - r\|a - u_r\|^2 + \frac{1}{2}r(\|a\|^2 + \|a - u_r\|^2) \\ &\le K(h) + \frac{1}{2}r\|a\|^2.\end{aligned}$$

Hence $[h, u_r - a]_+$ is bounded as $r \to 0$ by a constant depending on h. Then $\{u_r\}$ is bounded as $r \to 0$ by uniform boundedness principle. Since $f - ru_r \in Au_r$ and $ru_r \to 0$ as $r \to 0$, by Corollary 1, there exists $u \in D(A)$ such that $f \in Au$, that is, $f \in R(A)$

Now we establish a theorem for the range of sums of accretive operators, which satisfy

$$\sup_{[z,w]\in A} [w - f, y - z]_+ < \infty \tag{5}$$

for all $f \in R(A)$ and $y \in D(A)$.

For example if A is accretive and coercive in the sense that

$$\lim_{\substack{\|z\|\to\infty \\ w\in Az}} \frac{[w, z - y]_-}{\|z\|} = \infty$$

for each $y \in D(A)$, then it satisfies (5). For two subsets D_1 and D_2 of E we write $D_1 \simeq D_2$ if $\mathrm{cl}(D_1) = \mathrm{cl}(D_2)$ and $\mathrm{int}(D_1) = \mathrm{int}(D_2)$.

THEOREM 3. *Let E be a Banach space for which each nonempty bounded closed convex subset has the fixed point property for nonexpansive self-mappings. Let A and B be two accretive operator on E that satisfy the condition (5). If $A+B$ is m-accretive, then $R(A+B) \simeq R(A) + R(B)$.*

Proof. It is clear that $R(A + B) \subset R(A) + R(B)$. Let $F = R(A) + R(B)$. In order to apply Lemma 1 to $A + B$ and F, it is necessary to verify the condition (4). In fact, let $a \in D(A) \cap D(B)$ be fixed and let $f \in R(A) + R(B)$, that is, $f = f_1 + f_2$ with $f_1 \in R(A)$ and $f_2 \in R(B)$. Then, by the condition (5), we have

$$\sup_{[z,w_1]\in A} [w_1 - f_1, a - z]_+ < \infty$$

and

$$\sup_{[z,w_2]\in A} [w_2 - f_2, a - z]_+ < \infty.$$

This gives that $\sup_{[z,w]\in A+B}[w-f, a-z]_+ < \infty$. Thus $F \subset \mathrm{cl}(R(A+B))$ and $\mathrm{int}(R(A) + R(B)) \subset R(A + B)$ by Lemma 3. This completes the proof.

Theorem 3 asserts that Theorem 1.4 in [12] is true without the assumptions of smoothness and reflexivity of the space E. Theorem 3 also applies to all reflexive Banach spaces which possess normal structure.

DEFINITION 1. A Banach space E is said to have Property (I) if for any sequence $\{x_n\} \subset E$ and any sequence of positive numbers $\{\alpha_n\}$ with $\alpha_n \to 0$ as $n \to \infty$ satisfying

(i) $\alpha_n x_n \to 0$, *and*

(ii) *for each $h \in B_\rho(0)$ there exists $a(h) \in E$ and a constant$C(h)$ such that*

$$[h, x_n - a(h)]_+ \leq C(h),$$

it follows that $\{x_n\}$ is a bounded sequence (cf. [13]).

REMARK 1. Any Banach space whose duality mapping is uniformly continous on bounded sets satisfies Property (I)(cf. [1]).

LEMMA 4. *Let E be a Banach space which satisfies Property (I), $A \subset E \times E$ an m-accretive operator and F a subset of E. If for each $f \in F$ there is $a(f) \in E$ such that condition (4) holds, then $f \in \mathrm{cl}(R(A))$. If we assume further that each nonempty bounded closed* *convex subset of E has the fixed point property for nonexpansive self-mappings, then* $\text{int}(F) \subset R(A)$.

Proof. Let $f \in F$. For each $r > 0$, let $u_r \in E$ be such that $f \in Au_r + ru_r$. Then, using the notation of the proof of Lemma 3, we have

$$r\|u_r - a(f)\|^2 \leq K(f) + \frac{1}{2}r(\|a(f)\|^2 + \|u_r - a(f)\|^2).$$

Consequently, $\{ru_r\} \to 0$ as $r \to 0$ and hence $f \in \text{cl}(R(A))$.

Next, suppose that f belongs to $\text{int}(F)$. For each $r > 0$, let $u_r \in E$ be such that $f \in Au_r + ru_r$. Now let $\rho > 0$ such that $\|h\| \leq \rho$ implies $f + h \in F$. Again using the proof of Lemma 3, we also have

$$[h, u_r - a(h)]_+ \leq K(h) + \frac{1}{2}r\|a(h)\|^2.$$

Thus $\{u_r\}$ is bounded by Property (I) and the result follows from Corollary 1.

We close this section with the following result which slightly complements Theorem 1.7 in [12].

THEOREM 4. *Let E be a smooth Banach space which satisfies Property (I), and suppose that each nonempty bounded closed convex subset of E has the fixed point property for nonexpansive self-mappings. Let A and B be two accretive operators in E such that $D(A) \subset D(B)$ and satisfies condition (5). If $A+B$ is m-accretive, then $R(A)+R(B) \simeq R(A+B)$.*

Proof. It is clear that $R(A + B) \subset R(A) + R(B)$. Let $F = R(A) + R(B)$. In order to apply Lemma 4 to $A + B$ and F, it is necessary to verify the condition (4). In fact, let $f \in R(A)+R(B)$, that is, $f = f_1+f_2$ with $f_1 \in R(A)$ and $f_2 \in R(B)$ and let $a = a(f) \in D(A) \subset D(B)$ such that $f_1 \in Aa$ and $f_2 \in Ba$. By accretiveness of A, we have

$$(w_1 - f_1, J(a - z)) \leq 0$$

for all $[z, w_1] \in A$. On the other hand, by condition (5), there exists K such that

$$(w_2 - f_2, J(a - z)) \leq K$$

for all $[z, w_2] \in B$. This gives that $\sup_{[z,w] \in A+B}(w - f, J(a - z)) < +\infty$. Thus the result follows from Lemma 4.

REMARK 2. Theorem 4 applies to all uniformly smooth Banach spaces. In the case that E is a uniformly smooth Banach space, several related results are given in [9] and [12]. Interesting applications of related results are given in [5].

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