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SPECTRAL ANALYSIS OF JACOBI POLYNOMIALS IN KREIN SPACE

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1. Introduction

The Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ are polynomial solutions of the second order Strum-Liouville differential equation of the form

$$(x^2-1)y''+[(\alpha+\beta+2)x+(\alpha-\beta)]y' = n(n+\alpha+\beta+1)y, \qquad n=0,1,\dots,$$

where α, β, and $\alpha+\beta+1$ are not negative integers, and they are orthogonal with respect to the distributional weight

$$\omega(x) = (1-x)_+^{\alpha}(1+x)_+^{\beta}. \tag{1.1}$$

When α, $\beta > -1$, $\omega(x)$ is a locally integrable function given by

$$\omega(x) = \begin{cases} (1-x)^{\alpha}(1+x)^{\beta}, & -1 \le x \le 1, \\ 0, & \text{otherwise}. \end{cases} \tag{1.2}$$

When α or $\beta < -1$, a regularization of $(1-x)^{\alpha}(1+x)^{\beta}$ is required [7]. If $-N-1 < \alpha < -N$, $-M-1 < \beta < -M$, where N and M are positive integers, we have for any smooth function $\phi(x)$

$$\begin{aligned} \langle \omega, \phi \rangle = & \left(\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)2^{\alpha+\beta+1}} \right) \left(\int_0^1 (1-x)^{\alpha} \Big((1+x)^{\beta}\phi(x) \right. \\ & - \sum_{j=0}^{N-1} \frac{((1+x)^{\beta}\phi(x))^{j}}{j!} \Big|_{x=1} (-1)^j(1-x)^j \Big) dx \\ & + \int_{-1}^{0} (1+x)^{\beta} \Big((1-x)^{\alpha}\phi(x) \\ & - \sum_{k=0}^{M-1} \frac{((1-x)^{\alpha}\phi(x))^{(k)}}{k!} \Big|_{x=-1} (1+x)^k \Big) dx \end{aligned}$$

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$$+\sum_{j=0}^{N-1} \frac{((1+x)^\beta \phi(x))^{(j)}}{j!}\Big|_{x=1} \frac{(-1)^j}{(\alpha+1+j)}$$
$$+\sum_{k=0}^{M-1} \frac{((1-x)^\alpha \phi(x))^{(k)}}{k!}\Big|_{x=-1} \frac{1}{(\beta+1+k)}\Bigg).$$

For the so called classical orthogonal polynomial sets, i.e. Jacobi, Laguerre, Hermite, and Bessel polynomials, all of which satisfy second order Sturm-Liouville differential equation of the form (see [5]),

$$p(x)y'' + q(x)y' + r(x)y = \lambda y, \tag{1.4}$$

the spectral analysis of the associated Strum-Liouville problems is well advanced except for the Bessel polynomials and the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ when α or $\beta < -1$ [5].

Recently the spectral theory for the Bessel polynomials in a suitable Krein space is partially developed using a hyperfunctional weight. In case of Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}$ for α or $\beta < -1$, the regularization (1.3) generates an indefinite inner product $(f,g) = \langle \omega, f\bar{g}\rangle$, in contrast to the ordinary inner product

$$(f,g) = \int_{-1}^{1} f(x)\overline{g(x)}(1-x)^\alpha(1+x)^\beta\, dx, \quad \alpha,\beta > -1. \tag{1.5}$$

In this work, we construct an appropriate indefinite inner product space (which turns out to be a Krein space) in which the Jacobi operator

$$\ell = (x^2-1)D^2 + [(\alpha+\beta+2)x + (\alpha-\beta)]D \tag{1.6}$$

is self-adjoint and the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}$ are complete with real discrete eigenvalues $n(n+\alpha+\beta+1)$, $n = 0, 1, \ldots$.

Since the analysis for distinct pairs of α and β are essentially the same, we study here the special case $-N-1 < \alpha,\ \beta < -N$ and $-2N-1 < \alpha+\beta+1 < -2N$, N is an even integer.

2. Spectral Analysis

In [7] it was shown that the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ are orthogonal with respect to the sesquilinear form

$$(f,g) = \langle \omega, f\bar{g} \rangle, \tag{2.1}$$

and

$$\begin{aligned}(2.2)\quad &(P_n^{(\alpha,\beta)}(x), P_m^{(\alpha,\beta)}(x))\\ &= \frac{\Gamma(\alpha+\beta+2)\Gamma(\alpha+1+n)\Gamma(\beta+1+n)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+1+n)\Gamma(\alpha+\beta+1+2n)\,n!}\delta_{mn},\\ &m,n \geq 0\end{aligned}$$

where δ_{mn} is the Kroneker delta.

It naturally leads us to define spaces $J^{\pm}$ to be the span of

$$\{P_n^{(\alpha,\beta)} \mid (P_n^{(\alpha,\beta)},) > 0\} \text{ and } \{P_n^{(\alpha,\beta)} \mid (P_n^{(\alpha,\beta)}, P_n^{(\alpha,\beta)}) < 0\},$$

respectively.

Precisely, when $N = 2M$, J^+ is spanned by

$$\begin{aligned}&P_{2k+1}^{(\alpha,\beta)}(x), \quad k = 0,1,\dots,M-1,\\ &P_{2k}^{(\alpha,\beta)}(x), \quad k = M+1, M+2, \dots, 2M,\end{aligned}$$

and

$$P_n^{(\alpha,\beta)}(x), \quad n = 4M+1, 4M+2, \dots .$$

J^- is spanned by

$$P_{2k}^{(\alpha,\beta)}(x), \quad k = 0,1,\dots,2M,$$

and

$$P_{2k+1}^{(\alpha,\beta)}(x), \quad k = M+1, M+2, \ldots, 2M-1.$$

On the space $J = J^+ \oplus J^-$, define a new positive definite inner product by

$$[f,g] = \sum_0^\infty (f, e_n^{(\alpha,\beta)})(e_n^{(\alpha,\beta)}, g) \tag{2.3}$$

where

$$e_n^{(\alpha,\beta)}(x) = |(P_n^{(\alpha,\beta)}, P_n^{(\alpha,\beta)})|^{\frac{-1}{2}} P_n^{(\alpha,\beta)}(x)$$

is the normalization of $P_n^{(\alpha,\beta)}(x)$. Then $[f,g] = (f,g)$ on J^+ and $[f,g] = |(f,g)|$ on J^- and the space J with $[\cdot,\cdot]$ becomes a pre-Hilbert space. Let $K^\pm$ be the completions of $J^\pm$ respectively with respect to $[\cdot,\cdot]$ and let $K = K^+ \oplus K^-$.

We note that

$$K^+ = \left\{ \sum_0^\infty c_n e_n^{(\alpha,\beta)} \mid e_n^{(\alpha,\beta)} \in J^+, \sum_0^\infty |c_n|^2 < \infty \right\},$$

$$K^- = \left\{ \sum_0^\infty c_n e_n^{(\alpha,\beta)} \mid e_n^{(\alpha,\beta)} \in J^-, \sum_0^\infty |c_n|^2 < \infty \right\},$$

and

$$K = \left\{ \sum_0^\infty c_n e_n^{(\alpha,\beta)} \mid \sum_0^\infty |c_n|^2 < \infty \right\}.$$

Moreover, the space K with the indefinite inner product $(\cdot,\cdot)$ (resp. $[\cdot,\cdot]$) is a Krein space (resp. a Hilbert space). For more details on Krein spaces, we refer to Bognar [1].

The Jacobi operator ℓ in (1.6) is densely defined symmetric operator in K with domain J given by

$$\ell f = \sum_0^N n(n+\alpha+\beta+1) c_n e_n^{(\alpha,\beta)}, \tag{2.4}$$

for any $f = \sum_{0}^{N} c_n e_n^{(\alpha,\beta)}$ in J.

Define another linear operator L in K by

$$Lf = \sum_{0}^{\infty} n(n+\alpha+\beta+1) c_n e_n^{(\alpha,\beta)}, \tag{2.5}$$

for any $f = \sum_{0}^{\infty} c_n e_n^{(\alpha,\beta)}$ in D, where

$$D = \left\{ \sum_{0}^{\infty} c_n e_n^{(\alpha,\beta)} \in K \mid \sum_{0}^{\infty} |n(n+\alpha+\beta+1)|^2 < \infty \right\}.$$

Since $J \subset D$, L is also densely defined in K.

LEMMA 1. *L is symmetric, that is, for any f and g in D*

$$(Lf, g) = (f, Lg). \tag{2.6}$$

Proof. This is straightforward.

LEMMA 2. *For any complex number $\lambda \neq n(n+\alpha+\beta+1)$, $n \geq 0$ integer, $(L-\lambda I)D = K$ and $L - \lambda I$ has an inverse which is bounded with respect to $[\cdot,\cdot]$ and so the operator L is self-adjoint in K.*

Proof. For any $g = \sum_{0}^{\infty} d_n e_n^{(\alpha,\beta)}$ in K, consider $(L-\lambda I)f = g$. If $f = \sum_{0}^{\infty} c_n e_n^{(\alpha,\beta)}$, then we have

$$c_n = \frac{d_n}{n(n+\alpha+\beta+1) - \lambda}, \qquad n \geq 0. \tag{2.7}$$

For any $\lambda \neq n(n+\alpha+\beta+1)$, $n \geq 0$, both $\sum_{0}^{\infty} |c_n|^2$ and $\sum_0^\infty |n(n+\alpha+\beta+1)c_n|^2$ are finite so that f must be in D. Moreover, if we set $\frac{1}{\varepsilon} = \inf_{n\geq 0} |\lambda - n(n+\alpha+\beta+1)| > 0$, then $\sum_{0}^{\infty} |c_n|^2 \leq \varepsilon^2 \sum_{0}^{\infty} |d_n|^2$, so that $(L-\lambda I)^{-1}$ is bounded and for such real λ, $(L-\lambda I)^{-1}$ is self-adjoint since it is defined on the whole space K. Thus L is also self-adjoint.

As a consequence of Lemma 1 and Lemma 2, we now have

THEOREM 3. (a) *L is a self-adjoint extension of ℓ in K.*

(b) *The spectrum of L consists of only eigenvalues $n(n+\alpha+\beta+1)$, $n \geq 0$, with multiplicity 1 and Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}$ as corresponding eigenfunctions.*

Proof. (a) This is immediate consequence of Lemma 2.

(b) Since $LP_n^{(\alpha,\beta)}(x) = n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)$, $n \geq 0$ and there are no other points in the spectrum of L, for if $\lambda \neq n(n+\alpha+\beta+1)$, $n \geq 0$ integer, then $(L - \lambda I)$ is invertible and its range coincides with K, hence the spectrum of L consists of only eigenvalues $n(n+\alpha+\beta+1)$, $n \geq 0$, with single corresponding eigenfunction $P_n^{(\alpha,\beta)}$.

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