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THE FIRST EIGENVALUE ESTIMATE ON A COMPACT RIEMANNIAN MANIFOLD

BANG OK KIM

§1. Introduction

Let M be an n-dimensional compact Riemannian manifold with boundary ∂M. We consider the Neumann eigenvalue problem on M of the equation

$$\begin{aligned} \Delta u &= -\eta u \quad \text{in} \quad M \\ \frac{\partial u}{\partial \nu} &\equiv 0 \quad \text{on} \quad \partial M \end{aligned} \tag{1.1}$$

where ν is the unit outward normal vector to the boundary ∂M. Due to the importance of Poincaré inequality for analysis on manifolds, one wishes to obtain the lower bound of the first non-zero eigenvalue η_1 of (1.1). For the purpose of applications, it is important to relax the dependency of the lower bound on the geometric quantities. For general compact manifolds with convex boundary, Li-Yau [3] obtained the lower bound of η_1. Recently, Roger Chen [1] investigated the lower bound of the first Neumann eigenvalue η_1 on compact manifold M with non-convex boundary. We investigate the lower bound η_1 with the same conditions as those of Roger chen. But, using the different auxiliary function , we have the following theorem.

THEOREM 1.1. *Let M be an n-dimensional compact Riemannian manifold with boundary ∂M. Let ∂M satisfy the "interior rolling ε-ball" condition. Let R and H be positive constants such that the Ricci curvature of M is bounded below by $-R$ and the second fundamental*

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form elements of ∂M is bounded below by $-H$. If u is a solution of the equation (1.1) with $\eta = \eta_1$, then

$$\eta_1 \geq \frac{1}{\sqrt{1+\varepsilon H}} \frac{1-\alpha^2}{d^2(n-1)}(1+B)\exp(-(1+B)),$$

where

$$B = [1 + \frac{d^2(n-1)C}{1-\alpha^2}]^{\frac{1}{2}},$$
$$C = \frac{(2n-3)^2 + \alpha^2(10n-11)}{16\alpha^2(n-1)} 25H^2 + 2(1+\varepsilon H)^{\frac{1}{2}} R$$
$$+ \frac{25H}{4\varepsilon} + \frac{5}{2}(n-1)H(1+3H), \quad 0 < \alpha \leq \frac{1}{2},$$

and d is the diameter of M and the upper bound of ε is given by (2.8)and(2.9).

REMARK. R. Chen's result is given by $\eta_1 \geq \frac{1}{(1+H)^2}\frac{1-\alpha^2}{2(n-1)d^2}(1+B)exp(-(1+B))$, where α and ε are positive constants less than 1,

$$d = \text{diameter of } M,$$
$$B = [1 + \frac{4(n-1)d^2C}{1-\alpha^2}]^{\frac{1}{2}},$$
$$C = (1+H)C_1 + \frac{[(2n-3)^2 + (4n-5)\alpha^2]H^2}{(n-1)\varepsilon^2\alpha^2} + (1+H)^2 R,$$
$$C_1 = \frac{2(n-1)H(3H+1)(H+1)}{\varepsilon} + \frac{H+H^2}{\varepsilon^2}.$$

DEFINITION. Let ∂M be the boundary of a compact Riemannian manifold M. Then ∂M satisfies the "interior rolling ε-ball" condition if for each point $p \in \partial M$, there is a geodesic ball $B_q(\frac{\varepsilon}{2})$, centered at $q \in M$ with radius $\frac{\varepsilon}{2}$, such that $p = \overline{B_q(\frac{\varepsilon}{2})} \cap \partial M$ and $B_q(\frac{\varepsilon}{2}) \subset M$.

In §2, we shall give a gradient estimate which is essential in the proof of the main result.In §3, we shall give the proof of theorem1.1.

§2. A gradient estimate

THEOREM 2.1. *Let M be an n-dimensional compact Riemannian manifold with boundary ∂M satisfying the "interior rolling ε-ball" condition. Let R and H be positive constants such that the Ricci curvature of M is bounded below by $-R$ and the second fundamental form elements of ∂M is bounded below by $-H$. Let u be a solution of equation*

$$\Delta u + \eta_1 u = 0 \quad in \quad M$$
$$\frac{\partial u}{\partial \nu} \equiv 0 \quad on \quad \partial M$$

where ν is the unit outward normal vector to ∂M. Then

$$\frac{|\nabla u|^2}{(\beta - u)^2} \leq \max\left\{\frac{(n-1)}{1-\alpha^2}\left(C + \frac{2\beta}{\beta - \sup u}(1+\varepsilon H)^{\frac{1}{2}}\eta_1\right), \frac{\sqrt{2}}{\sqrt{1-\alpha^2}}\frac{\sqrt{1+\varepsilon H}\,\eta_1 \sup u}{(\beta - \sup u)}\right\},$$

where

$$C = \frac{(2n-3)^2 + \alpha^2(10n-11)}{16\alpha^2(n-1)}25H^2 + 2(1+\varepsilon H)^{\frac{1}{2}}R + \frac{25H}{4\varepsilon} + \frac{5}{2}(n-1)H(1+3H), \quad 0 < \alpha \leq \frac{1}{2}, \quad \beta > \sup u \quad .$$

Proof. Let $\psi(r)$ be a nonnegative C^2-function defined on $[0,\infty)$ such that

$$\psi(r) = \begin{cases} \leq \varepsilon H & \text{if} \quad r \in [0, \varepsilon) \\ = \varepsilon H & \text{if} \quad r \in [\varepsilon, \infty) \end{cases}$$

with $\psi(0) = 0$, $\psi'(0) = 5H$, $\psi'' \geq -\frac{25H}{2\varepsilon}$ and $5H \geq \psi'(r) \geq 0$.

Define $\phi(x) = \psi(r(x))$, where $r(x)$ denotes the distance function from boundary ∂M to $x \in M$. Let $\beta > \sup u$.

We define the auxiliary function $G(x) = (1+\phi)^{\frac{1}{2}} \dfrac{|\nabla u|^2}{(\beta - u)^2}$. By the compactness of M, there is a point $x_0 \in M$ such that G achieves its supremum.

Suppose that x_0 is a boundary point of ∂M. At x_0, we may choose an orthonormal frame field $e_1, e_2, \ldots, e_n$ such that $e_n = \frac{\partial}{\partial \nu}$.
Then we have, $\frac{\partial G}{\partial \nu}(x_0) \geq 0$. This gives

$$0 \leq \frac{1}{2}\frac{\partial \phi}{\partial \nu}\frac{|\nabla u|^2}{(\beta - u)^2} + \frac{2\sum_{i=1}^{n} u_i u_{i\nu}}{(\beta - u)^2}$$

If h_{ij} are the second fundamental form elements of ∂M, we see that

$$\begin{aligned} u_{i\nu} &= e_i e_n u - (\nabla_{e_i} e_n) u, \\ &= -\sum_{j=1}^{n-1} h_{ij} u_j, \text{ for } 1 \leq i \leq n-1. \end{aligned}$$

Hence we obtain

$$0 \leq \frac{\partial G}{\partial \nu}(x_0) \leq \frac{1}{2}\frac{|\nabla u|^2}{(\beta - u)^2}(-5H + 4H) < 0,$$

which is a contradiction. Therefore x_0 has to be an interior point of M. Hence $\nabla G(x_0) = 0$ and $\Delta G(x_0) \leq 0$.

At x_0, we may choose an orthonormal frame field $\{e_i\}$ such that $u_1(x_0) = |\nabla u(x_0)|$. Since, for each i, $G_i(x_0) = 0$, we obtain that

$$\begin{aligned} \text{(2.1)} \qquad & \text{if } i \neq 1 \quad u_{1i} = -\frac{1}{4}(1+\phi)^{-1}\phi_i |\nabla u|, \\ & \text{if } i = 1 \quad u_{11} = -\frac{|\nabla u|^2}{(\beta - u)} - \frac{1}{4}(1+\phi)^{-1}\phi_1 |\nabla u|. \end{aligned}$$

By using (2.1) and $u_{ijk} - u_{ikj} = \sum_{l=1}^{n} u_l R_{lijk}$, we have

(2.2)

$$\Delta G(x_0) = -\frac{1}{4}(1+\phi)^{-\frac{3}{2}}|\nabla\phi|^2\frac{|\nabla u|^2}{(\beta-u)^2} + \frac{1}{2}(1+\phi)^{-\frac{1}{2}}\Delta\phi\frac{|\nabla u|^2}{(\beta-u)^2}$$
$$+ 2(1+\phi)^{-\frac{1}{2}}\phi_1\frac{u_1}{(\beta-u)^2}\left(-\frac{|\nabla u|^2}{(\beta-u)} - \frac{1}{4}(1+\phi)^{-1}\phi_1|\nabla u|\right)$$
$$+ \sum_{i=2}^{n} 2(1+\phi)^{-\frac{1}{2}}\phi_i\frac{u_1}{(\beta-u)^2}\left(-\frac{1}{4}(1+\phi)^{-1}\phi_i|\nabla u|\right)$$
$$+ 2(1+\phi)^{-\frac{1}{2}}\phi_1\frac{|\nabla u|^3}{(\beta-u)^3} + \sum_{ij=1}^{n} 2(1+\phi)^{\frac{1}{2}}\frac{(u_{ji})^2}{(\beta-u)^2}$$
$$+ \sum_{j=1}^{n} 2\frac{(1+\phi)^{\frac{1}{2}}}{(\beta-u)^2}(u_j(\Delta u)_j) + \sum_{ij=1}^{n} 2\frac{(1+\phi)^{\frac{1}{2}}}{(\beta-u)^2}u_j u_i R_{ij}$$
$$+ \frac{8(1+\phi)^{\frac{1}{2}}}{(\beta-u)^3}|\nabla u|^2\left(-\frac{|\nabla u|^2}{(\beta-u)} - \frac{1}{4}(1+\phi)^{-1}\phi_1|\nabla u|\right)$$
$$+ 2(1+\phi)^{\frac{1}{2}}\left(\frac{-\eta_1 u|\nabla u|^2}{(\beta-u)^3}\right) + 6(1+\phi)^{\frac{1}{2}}\frac{|\nabla u|^4}{(\beta-u)^4}.$$

It is clear that

$$\sum_{ij=1}^{n} u_{ji}{}^2 \geq \sum_{i=1}^{n} u_{ii}{}^2 \geq u_{11}{}^2 + \frac{1}{n-1}(\Delta u - u_{11})^2 \tag{2.3}$$
$$\geq u_{11}{}^2 + \frac{u_{11}{}^2}{2(n-1)} - \frac{(\Delta u)^2}{n-1}.$$

Multiplying (2.2) by $(1+\phi)^{\frac{1}{2}}\frac{(\beta-u)^2}{|\nabla u|^2}$ and substituting (2.3), we have

$$0 \geq \frac{(1+\phi)}{(n-1)}\frac{|\nabla u|^2}{(\beta-u)^2} + \frac{-(2n-3)}{2(n-1)}\phi_1\frac{|\nabla u|}{(\beta-u)} \tag{2.4}$$
$$- \frac{2(1+\phi)}{(n-1)}\frac{\eta_1^2 u^2}{|\nabla u|^2} + \frac{(2n-1)}{16(n-1)}(1+\phi)^{-1}\phi_1^2$$

$$- \frac{2\eta_1 u(1+\phi)}{(\beta - u)} - 2R(1+\phi) - 2\eta_1(1+\phi)$$
$$- \frac{3}{4}(1+\phi)^{-1}|\nabla\phi|^2 + \frac{1}{2}\Delta\phi.$$

It is clear that

$$\frac{\alpha^2(1+\phi)}{(n-1)}\frac{|\nabla u|^2}{(\beta-u)^2} - \frac{(2n-3)}{2(n-1)}\phi_1\frac{|\nabla u|}{(\beta-u)} \geq -\frac{1}{16}\frac{(2n-3)^2}{\alpha^2(n-1)}(1+\phi)^{-1}\phi_1^2. \tag{2.5}$$

Substituting (2.5) into (2.4), we have, for $0 < \alpha < 1$,

$$\begin{aligned} 0 \geq & \frac{(1-\alpha^2)(1+\phi)}{(n-1)}\frac{|\nabla u|^2}{(\beta-u)^2} \\ & - \frac{1}{16\alpha^2}\frac{(2n-3)^2}{(n-1)}(1+\phi)^{-1}\phi_1^2 \\ & - \frac{2(1+\phi)\eta_1^2 u^2}{(n-1)|\nabla u|^2} + \frac{(2n-1)}{16(n-1)}(1+\phi)^{-1}\phi_1^2 \\ & - \frac{2\eta_1 u}{(\beta-u)}(1+\phi) - 2(1+\phi)(R+\eta_1) \\ & - \frac{3}{4}(1+\phi)^{-1}|\nabla\phi|^2 + \frac{1}{2}\Delta\phi. \end{aligned} \tag{2.6}$$

Multiplying (2.6) by $\frac{|\nabla u|^2}{(\beta-u)^2}$, we obtain

(2.7)

$$\begin{aligned} 0 \geq & \frac{(1-\alpha^2)}{(n-1)}G(x_0)^2 \\ & - G(x_0)\Big\{\frac{(2n-3)^2+\alpha^2(10n-11)}{16\alpha^2(n-1)}(1+\phi)^{-\frac{3}{2}}|\nabla\phi|^2 \\ & + 2(1+\phi)^{\frac{1}{2}}(R+\eta_1) + \frac{2\eta_1 u}{(\beta-u)}(1+\phi)^{\frac{1}{2}} - \frac{1}{2}\Delta\phi(1+\phi)^{-\frac{1}{2}}\Big\} \\ & - \frac{2(1+\phi)\eta_1^2 u^2}{(\beta-u)^2(n-1)}, \quad \text{for} \quad 0 < \alpha \leq \frac{1}{2}. \end{aligned}$$

To compute $\Delta\phi$, let $\partial M(\varepsilon) = \{x \in M \mid r(x) \leq \varepsilon\}$ and k_ε be the upper bound of the sectional curvature in $\partial M(\varepsilon)$. we may choose ε to be small so that,

$$\sqrt{k_\varepsilon} tan(\varepsilon\sqrt{k_\varepsilon}) \leq \frac{H}{2} + \frac{1}{2} \tag{2.8}$$

$$\frac{H}{\sqrt{k_\varepsilon}} tan(\varepsilon\sqrt{k_\varepsilon}) \leq \frac{1}{2}. \tag{2.9}$$

By using an index comparison theorem in Riemannian geometry [4], one can show that if $x \in \partial M(\varepsilon)$, we have

$$\Delta r \geq -(n-1)\frac{H + \sqrt{k_\varepsilon} tan(\varepsilon\sqrt{k_\varepsilon})}{1 - tan(\varepsilon\sqrt{k_\varepsilon})H/\sqrt{k_\varepsilon}} \geq -(n-1)(3H+1).$$

Then we have

$$\begin{aligned}\Delta\phi &= \psi''|\nabla r|^2 + \psi'\Delta r \\ &\geq -\frac{25H}{2\varepsilon} - \frac{5}{2}(n-1)H(1+3H).\end{aligned}$$

From now, let

$$C_1 = \frac{25H}{4\varepsilon} + \frac{5}{2}(n-1)H(1+3H).$$

Hence, (2.7) becomes

(2.10)

$$\begin{aligned}0 \geq &\frac{(1-\alpha^2)}{n-1}G(x_0)^2 - G(x_0)\Big\{\frac{(2n-3)^2 + \alpha^2(10n-11)}{16\alpha^2(n-1)}25H^2 \\ &+ 2(1+\varepsilon H)^{\frac{1}{2}}R + \frac{2\beta}{\beta - \sup u}(1+\varepsilon H)^{\frac{1}{2}}\eta_1 + C_1\Big\} \\ &- \frac{2(1+\varepsilon H)\eta_1^2(\sup u)^2}{(n-1)(\beta - \sup u)^2}.\end{aligned}$$

Thus (2.7) is a quadratic equation for $G(x_0)$.

Hence

$$\begin{aligned}\frac{|\nabla u|^2}{(\beta-u)^2}(x) \leq & G(x_0)\\ \leq & \max\Big[\frac{(n-1)}{(1-\alpha^2)}\Big\{\frac{(2n-3)^2+\alpha^2(10n-11)}{16\alpha^2(n-1)}25H^2\\ & +2(1+\varepsilon H)^{\frac{1}{2}}R+C_1\\ & +\frac{2\beta}{\beta-\sup u}(1+\varepsilon H)^{\frac{1}{2}}\eta_1\Big\},\ \frac{\sqrt{2}\sqrt{1+\varepsilon H}\eta_1 \sup u}{\sqrt{1-\alpha^2}(\beta-\sup u)}\Big].\end{aligned}$$

REMARK. By the fact that $\frac{H}{\sqrt{k_\varepsilon}}tan(\varepsilon\sqrt{k_\varepsilon}) \leq \frac{1}{2}$, we can choose a geodesic from boundary to x_0 which has no focal point. Hence we can use the index comparison theorem.

REMARK. The "interior rolling ε-ball" condition is necessary for η_1 being bounded away from 0, see [1].

§3. Proof of Theorem 1.1

Let u be a first eigenfunction of (1.1). We can assume that $\sup u = 1$ and $\inf u = -k \geq -1$.

Proof. From Theorem 2.1, we know that

(3.1)

$$\begin{aligned}\frac{|\nabla u|}{(\beta-u)}(x) \leq & \Big(\frac{n-1}{1-\alpha^2}\Big)^{\frac{1}{2}}\Big[\frac{(2n-3)^2+\alpha^2(10n-11)}{16\alpha^2(n-1)}25H^2\\ & +2(1+\varepsilon H)^{\frac{1}{2}}R+C_1+\frac{2\beta}{\beta-1}(1+\varepsilon H)^{\frac{1}{2}}\eta_1\Big]^{\frac{1}{2}}.\end{aligned}$$

Since u satisfies $\int_M u = 0$ and $u \not\equiv 0$, the nodal set N of u divides $\overline{M}$ into two parts. If $x_M \in \overline{M}$ is a point where u achieves its supremum and γ is the shortest geodesic joining x_M and N, then γ has length at most diameter d of M.

Integrating (3.1) along γ, we have

$$\log \frac{\beta}{\beta-1} \le \int_\gamma \frac{|\nabla u|}{\beta - u}$$
$$\le \left(\frac{n-1}{1-\alpha^2}\right)^{\frac{1}{2}} \Big[\frac{(2n-3)^2+\alpha^2(10n-11)}{16\alpha^2(n-1)} 25H^2$$
$$+ 2(1+\varepsilon H)^{\frac{1}{2}} R + C_1 + \frac{2\beta}{\beta-1}(1+\varepsilon H)^{\frac{1}{2}}\eta_1\Big]^{\frac{1}{2}} d.$$

Thus

$$\eta_1 \ge \frac{1}{2\sqrt{1+\varepsilon H}} \frac{\beta-1}{\beta}\Big\{\frac{(1-\alpha^2)}{(n-1)d^2}(\log\frac{\beta}{\beta-1})^2 - C\Big\}, \tag{3.2}$$

where

$$C = \frac{(2n-3)^2+\alpha^2(10n-11)}{16\alpha^2(n-1)} 25H^2 + 2(1+\varepsilon H)^{\frac{1}{2}} R + C_1.$$

Let

$$f(\beta) = \frac{1}{2\sqrt{1+\varepsilon H}} \frac{\beta-1}{\beta}\Big\{\frac{(1-\alpha^2)}{(n-1)d^2}(\log\frac{\beta}{\beta-1})^2 - C\Big\}.$$

Then $f(\beta)$ has a maximum at

$$\frac{\beta}{\beta-1} = e^{1+\sqrt{1+\frac{(n-1)d^2}{1-\alpha^2}C}}. \tag{3.3}$$

Substituting (3.3) into (3.2), we have that

$$\eta_1 \ge \frac{1}{\sqrt{1+\varepsilon H}} \frac{(1-\alpha^2)}{(n-1)d^2}(1+B)e^{-(1+B)},$$

where

$$B = [1 + \frac{(n-1)d^2}{(1-\alpha^2)} C]^{\frac{1}{2}}$$
$$C = \frac{(2n-3)^2+\alpha^2(10n-11)}{16\alpha^2(n-1)} 25H^2 + 2(1+\varepsilon H)^{\frac{1}{2}} R$$
$$+ \frac{25H}{4\varepsilon} + \frac{5}{2}(n-1)H(1+3H).$$

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DEPARTMENT OF MATHEMATICS, SEOUL NATIONAL UNIVERSITY, SEOUL 151-742, KOREA