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FIXED POINT ITERATIONS FOR QUASI–CONTRACTIVE MAPS IN UNIFORMLY SMOOTH BANACH SPACES

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1. Introduction

Suppose K is a nonempty subset of a normed linear space E, and T is a mapping of K into itself. Then T is called *quasi-contractive* (see e.g., [8]) if there exists a constant $k \in [0,1)$ such that,

$$(1)\quad \|Tx-Ty\| \leq k \max\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \|x-Ty\|, \|y-Tx\|\}$$

for all x, y in K. In [19], Rhoades showed that the contractive definition (1), apart from being an obvious generalization of the well-known contraction mapping, is one of the most general contractive definitions for which Picard iterations give a unique fixed point. In [18], Rhoades examined the following two fixed point iteration schemes:

(a) *The Ishikawa Iteration Process*: (see e g., [10], [18]) defined as follows: For K a convex subset of a Banach space E and T a mapping of K into itself, the sequence $\{x_n\}_{n=0}^{\infty}$ in K is defined by:

$$(2)\qquad x_0 \in K$$

$$(3)\qquad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n$$

$$(4)\qquad y_n = (1-\beta_n)x_n + \beta_n T x_n,\ n \geq 0$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ satisfy:

(i) $0 \leq \alpha_n \leq \beta_n < 1,\ n \geq 0,$
(ii) $\lim_{n\to\infty} \beta_n = 0$, and
(iii) $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$

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the map, to the unique fixed point of a quasi-contractive mapping in L_p (or ℓ_p), spaces, $p \in [2, \infty)$, and in [7] he extended these results to L_p (or ℓ_p), spaces, $p \in (2, \infty]$. Some authors (see. e.g., [15], [16]) probably unaware of these results of Chidume, have recently answered the above question in the affirmative *in Hilbert spaces*, proving in the process results which are special cases of the results of [4], [6] and [7].

More recently, Qihou [15] considered iterates of quasi-contractive mappings still *in Hilbert spaces* and proved the following theorem:

THEOREM LQ ([15], P.302). *Let T be a quasi-contractive mapping in a bounded closed convex subset K of a Hilbert space and let $\{x_n\}_{n=0}^{\infty}$, and $\{\beta_n\}_{n=0}^{\infty}$ satisfy:*

(i) $0 \leq \beta_n \leq 1; \quad n \geq 0$
(ii) $\lim_{n\to\infty} \beta_n = 0$;
(iii) $\frac{1-k^2}{2} \leq \alpha_n \leq 1 - k^2$.

Then, for each $x_0 \in K$, the sequence of Ishikawa iterates converges to the unique fixed point of T.

While the existence of fixed points for quasi-contractive maps has been proved in general Banach spaces (see e.g., [8], [19]), iterative methods for approximating such fixed points have largely been confined to Hilbert spaces (see e.g., [15], [16], [18]).

It is known that among all Banach spaces, Hilbert spaces are the ones with the best geometric structures (see e.g., [12], [25]) in the sense that certain geometric structures which characterize Hilbert spaces make the solution of problems posed in such spaces relatively straightforward. Consequently, to extend solutions of such problems to general Banach spaces the Banach spaces should be characterized by relations similar to those that characterize Hilbert spaces. Several authors are now conduction worthwhile research in this direction (see e.g., [1], [2], [3], [4-7], [9], [11], [14], [17], [21], [22-25]).

Recently, Hong-Kun Xu [23] studied *uniformly smooth Banach spaces with modulus of smoothness of power type $q > 1$* (defined below). These spaces include the L_p (or ℓ_p), W_m^p and e^p spaces, $1 < p < \infty$. Xu [23] obtained several inequalities in this space which generalize some known L_p inequalities:

For $q > 1$ we shall denote by J_q the generalized duality mapping from E to 2^{E^*} given by

$$J_q(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^q \text{ and } \|f^*\| = \|x\|^{q-1}\}.$$

In particular $J = J_2$ is called the normalized duality map on E. It is well-known (see e.g., [25]) that E is uniformly smooth if and only if J (and hence J_q) is single valued and uniformly continuous on any bounded subset of E.

In his study of characteristic inequalities of uniformly smooth Banach spaces with modulus of smoothness of power type $q > 1$, Xu [23] proved the following theorem:

Theorem HKS ([23], Corollary 1', P.1130): Let E be a uniformly smooth Banach space. Then E has modulus of smoothness of power type $q > 1$ if and only if there exists a constant $c > 0$ such that:

$$\|x + y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + c\|y\|^q \tag{7}$$

for all x, y in E.

For Hilbert spaces $q = 2, c = 1$ and equality holds. For $p \geq 2, L_p\,(\text{or } \ell_p)$ spaces have modulus of smoothness of power type $q = 2$ and (7) is satisfied with $c = (p-1)$ (see e.g., [23]). Also $L_p\,(\text{or } \ell_p)$ spaces, $p \in (1, 2]$ have modulus of smoothness of power type $q = p$ and hence satisfy (7). For an estimate of the constant c for $L_p\,(\text{or } \ell_p)$, spaces $p \in (1, 2]$ the reader may consult [23].

In the sequel we shall need the following result:

LEMMA LQ ([15], P.302). *Let $\{x_n\}_{n=1}^{\infty}$ satisfy the following inequality:*

$$x_{n+1} \le \omega x_n + \sigma_n, \quad n \ge 1$$

where $x_n \geq 0, \sigma_n \geq 0$ and $\lim_{n\to\infty} \sigma_n = 0, 0 \le \omega < 1$. Then $x_n \to 0$ as $n \to \infty$.

REMARK 1. As has been observed, L_p (or ℓ_p), spaces $p \in [2, \infty)$ have modulus of smoothness of power type $q = 2$ and satisfy (7) for $c = p-1$. By setting $q = 2$ and $c = (p-1)$ in (8) we obtain Lemma (9) of [5], for L_p spaces, $p \in [2, \infty)$ (see also inequality (4) of [4]).

For the remainder of this paper, k will denote the constant appearing in definition of quasi-contractive map, and c will denote the constant appearing in our lemma. We prove the following theorems:

THEOREM 1. *Let E be a real uniformly smooth Banach space with modulus of smoothness of power type $q > 1$. Let K be a closed, convex and bounded subset of E, and $T : K \to K$ a quasi-contractive mapping of K into itself with $ck^q < min\{q-1, 1\}$. Let $\{c_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying;*

(i) $0 \leq \beta_n < 1, \quad n \geq 0$
(ii) $\lim_{n\to\infty} \beta_n = 0,$
(iii) $\frac{1}{2}[\frac{1}{c}(1-ck^q)]^{\frac{1}{q-1}} \leq \alpha_n \leq [\frac{1}{c}(1-ck^q)]^{\frac{1}{q-1}}.$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined iteratively by,

$$x_0 = K,$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n$$
$$y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 0$$

converges strongly to the unique fixed point of T.

Proof. From Ciric [8] and the conclusion of Rhoades [19], T has a unique fixed point x^* (say) in K. Set $y = x^*$ in (1) to obtain;

$$\|Tx - x^*\| \leq k \max\{\|x - x^*\|, \|x - Tx\|\}$$

for each $x \in K$. In particular,

$$\|Ty_n - x^*\| \leq k \max\{\|y_n - x^*\|, \|y_n - Ty_n\|\},$$

so that for all non-negative integers n, the following inequalities hold:

$$\|Ty_n - x^*\|^q \leq k^q(\|y_n - x^*\|^q + \|y_n - Ty_n\|^q) \tag{13}$$
$$\|Tx_n - x^*\|^q \leq k^q(\|x_n - x^*\|^q + \|x_n - Tx_n\|^q). \tag{14}$$

We can now estimate $\|x_{n+1} - x^*\|^q$ using (17)

$$
\begin{aligned}
\|x_{n+1} - x^*\|^q =& (1-\alpha_n)(x_n - x^*) + \alpha_n(Ty_n - x^*)\|^q \\
\leq& [1-\alpha_n(q-1)]\ \|x_n - x^*\|^q + c\alpha_n\|Ty_n - x^*\|^q \\
& - \alpha_n[1 - c\alpha_n^{q-1}]\ \|x_n - Ty_n|\ ^q \\
\leq& [1-\alpha_n(q-1-ck^q)]\ \|x_n - x^*\|^q \\
& - \alpha_n[1 - c\alpha_n^{q-1} - ck^q(1-\beta_n(q-1))]\ \|x_n - Ty_n\|^q \\
& + \alpha_n\beta_n cD
\end{aligned}
$$

i.e.,

(18)
$$
\begin{aligned}
\|x_{n+1} - x^*\|^q \leq& [1-\alpha_n(q-1-ck^q)]\ \|x_n - x^*\|^q \\
& - \alpha_n[1 - c\alpha_n^{q-1} - ck^q(1-\beta_n(q-1))]\|x_n - Ty_n\|^q \\
& + \alpha_n\beta_n M,
\end{aligned}
$$

where $M = cD$.

Condition (iii) implies;

$$1-\alpha_n(q-1-ck^q) \leq 1 - \frac{1}{2}[\frac{1}{c}(1-ck^q)]^{\frac{1}{q-1}}(q-1-ck^q) = h_q \quad \text{(say)} \quad < 1,$$

and,

$$1 - c\alpha_n^{q-1} - ck^q(1-\beta_n(q-1)) \geq 1 - c\alpha_n^{q-1} - ck^q \geq 0,$$

so that for sufficiently large n, inequality (18) reduces to;

$$\|x_{n+1} - x^*\|^q \leq h_q\|x_n - x^*\|^q + \alpha_n\beta_n M, \quad h_q \in (0,1).$$

Clearly, $\alpha_n\beta_n M \to \infty$ and by setting $\rho_n = \|x_n - x^*\|^q, \omega = h_q$, and $\sigma_n = \alpha_n\beta_n M$, it follows from lemma LQ that $x_n \to x^*$ as $n \to \infty$, completing the proof of Theorem 1.

Iteration of inequality (19) from $n = N_0$ to N yields:

$$\|x_{N+1} - x^*\|^q \leq \prod_{j=N_0}^{N} [1 - c_j(q - 1 - ck^q) \|x_{N_0} - x^*\|^q \to 0$$

as $N \to \infty$, by condition (iii). Hence $x_n \to x^*$ as $n \to \infty$, completing the proof of Theorem 2.

REMARK 3. Theorems 1 and 2 show that either the Mann iteration method or the Ishikawa iteration method can used to approximate the fixed point of a quasi-contractive map in *real uniformly smooth banach spaces with modulus of smoothness of power type* $q > 1$. However, the Mann iteration method may be preferred due to its simplicity.

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