Bull. Korean Math. Soc. 30 (1993), No. 2, pp. 193–199

FULL HEREDITARY C^*–SUBALGEBRAS OF CROSSED PRODUCTS

JA A JEONG

1. Introduction

A hereditary C^*-subalgebra B of a C^*-algebra A is said to be *full* if B is not contained in any proper closed two-sided ideal in A, so each hereditary C^*-subalgebra of a simple C^*-algebra is always full. It is well known that every C^*-algebra is strong Morita equivalent to its full hereditary C^*-subalgebra, but the strong Morita equivalence of a C^*-algebra A and its hereditary C^*-subalgebra B does not imply the fullness of B, in general. In [1], Combes found some conditions for C^*-dynamical systems under which two C^*-crossed products $A \underset{\alpha}{\times} G$ and $B \underset{\beta}{\times} G$ become strong Morita equivalent. In particular, from his result, we see that if B is an α-invariant full hereditary C^*-subalgebra of A and $\beta = \alpha|_B$ then $A \underset{\alpha}{\times} G$ is strong Morita equivalent to $B \underset{\beta}{\times} G$.

In this short note, we show that the hereditary ([2]) C^*-subalgebra $B \underset{\alpha}{\times} G$ (respectively $B \underset{\alpha r}{\times} G$) is actually full in $A \underset{\alpha}{\times} G$ (respectively $A \underset{\alpha r}{\times} G$) when G is a discrete group and B is full in A.

For a subset B of an involutive Banach algebra A and a discrete group G, we denote:

$$\begin{aligned} L_B &= \{a \in A \mid a^*a \in B\} \\ \mathbf{k}(G,B) &= \{\ k : G \to A \mid k(G) \subseteq B,\ \operatorname{supp} k \text{ is finite }\}. \end{aligned}$$

If A and B are C^*-algebras, then $B \subseteq L_B$. Furthermore, if B is hereditary in A, then L_B is a closed left ideal of A and ${L_B}^* L_B = B$. It will

Received June 1, 1992. Revised December 21, 1992.

1991 *Mathematics Subject Classification. 46L55.* *Supported by Global Analysis Research Center.*

To prove (2.2), it is enough to see that $\mathbf{k}(G, L_B{}^*) = \mathbf{k}(G, L_B)^*$ which is but almost obvious.

For any non-zero $a_1, a_2 \in L_B$, and $t_0 \in G$, define two functions $f \in \mathbf{k}(G, L_B)$ and $g \in \mathbf{k}(G, L_B{}^*)$ by

$$f(s) = \begin{cases} a_1, & s = t_0 \\ 0, & s \neq t_0 \end{cases} \quad \text{and} \quad g(s) = \begin{cases} \alpha_{t_0^{-1}}(a_2{}^*), & s = e \\ 0, & s \neq e. \end{cases}$$

Then

$$fg \in \mathbf{k}(G, L_B)\mathbf{k}(G, L_B{}^*) \subseteq L_{\mathbf{k}(G,B)} L_{\mathbf{k}(G,B)}{}^*$$

by (2.1), (2.2), and

$$\begin{aligned} (fg)(t) &= \sum_s f(s)\alpha_s(g(s^{-1}t)) \\ &= f(t_0)\alpha_{t_0}(g(t_0{}^{-1}t)) \\ &= \begin{cases} a_1 a_2{}^*, & t = t_0 \\ 0, & t \neq t_0. \end{cases} \end{aligned}$$

Hence, $fg \in \mathbf{k}(G, L_B L_B{}^*)$ and $\operatorname{supp}(fg) = \{t_0\}$. We proved that for any $a_1 a_2{}^* \in L_B L_B{}^*$, $t_0 \in G$, there exists $f \in \mathbf{k}(G, L_B)\mathbf{k}(G, L_B{}^*)$ such that $f(t_0) = a_1 a_2{}^*$ with $\operatorname{supp} f = \{t_0\}$. Since $L_B L_B{}^*$ is a two-sided ideal in A, each element of $L_B L_B{}^*$ is a linear sum of elements of the form $a_1 a_2{}^*$ for $a_1, a_2 \in L_B$, and hence

$$\begin{aligned} \mathbf{k}(G, L_B L_B{}^*) &\subseteq \mathbf{k}(G, L_B)\mathbf{k}(G, L_B{}^*) \\ &\subseteq L_{\mathbf{k}(G,B)} L_{\mathbf{k}(G,B)}{}^*. \end{aligned}$$

We refer to [4] (or [5]) for definition of (reduced) C^*-crossed product. For an α-invariant C^*-subalgebra B of A, although in general the crossed product $B \underset{\alpha}{\times} G$ need not be a C^*-subalgebra of $A \underset{\alpha}{\times} G$, hereditariness of B guarantees that, as was proved by Kusuda [2]: Let (A, G, α) be a C^*-dynamical system. If B is an α-invariant hereditary C^*-subalgebra of A, then $B \underset{\alpha}{\times} G$ is a hereditary C^*-subalgebra of $A \underset{\alpha}{\times} G$.

COROLLARY 2.3. *Let (A, G, α) be a C^*-dynamical system with a discrete group G. If A is simple then $A \underset{\alpha}{\times} G$ has no proper closed two-sided ideal which contains a hereditary C^*-subalgebra $B \underset{\alpha}{\times} G$ arising from an α-invariant hereditary C^*-subalgebra B of A.*

Concerning the reduced crossed product, we first have to show that $B \underset{\alpha r}{\times} G$ is a hereditary C^*-subalgebra of $A \underset{\alpha r}{\times} G$ whenever B is an α-invariant hereditary C^*-subalgebra of A. To show the hereditariness of a C^*-subalgebra, it would be a useful fact that if B is a C^*-subalgebra of a C^*-algebra A then B is hereditary in A if and only if $bab' \in B$ for all $b, b' \in B$ and $a \in A$ [3].

THEOREM 2.4. *Let (A, G, α) be a C^*-dynamical system and G be a discrete group. If B is an α-invariant full hereditary C^*-subalgebra of A then $B \underset{\alpha r}{\times} G$ is also full hereditary in $A \underset{\alpha r}{\times} G$.*

Proof. $B \underset{\alpha r}{\times} G$ is a C^*-subalgebra of $A \underset{\alpha r}{\times} G$ [4]. Recall that for any representation (π, H) of A, a representation $(\widetilde{\pi}, \ell^2(G, H))$ of A is defined by

$$(\widetilde{\pi}(a)\xi)(t) = \pi(\alpha_{t^{-1}}(a))\xi(t)$$

for every a in A, t in G and ξ in $\ell^2(G, H)$ [4]. To prove the hereditariness of $B \underset{\alpha r}{\times} G$ in $A \underset{\alpha r}{\times} G$, we show that for any $y = \sum_t \widetilde{\pi}_u(y_0(t))\lambda_t$ with $y_0 \in \mathbf{k}(G, A)$, we have

$$fyg \in B \underset{\alpha r}{\times} G$$

for all $f = \sum_s \widetilde{\pi}_u(f_0(s))\lambda_s$, $g = \sum_h \widetilde{\pi}_u(g_0(h))\lambda_h$ with $f_0, g_0 \in \mathbf{k}(G, B)$, where (π_u, H_u) is the universal representation of A and λ is the left regular representation of G on $\ell^2(G, H_u)$. In fact,

$$fyg = \sum_{s,t,h} \widetilde{\pi}_u(f(s)\alpha_s(y_0(t))\alpha_{st}(g_0(h)))\lambda_{sth}$$

and from the assumption that B is α-invariant hereditary in A, we see that

$$f(s)\alpha_s(y_0(t))\alpha_{st}(g_0(h)) \in B$$

4. G. K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, New York, 1979.
5. J. Tomiyama, *Invitation to C^*-algebras and topological dynamics*, World Scientific, Singapore, 1987.

GLOBAL ANALYSIS RESEARCH CENTER, DEPARTMENT OF MATHEMATICS, SEOUL NATIONAL UNIVERSITY, SEOUL 151-742, KOREA