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CONTINUITY OF JORDAN $*$–HOMOMORPHISMS OF BANACH $*$–ALGEBRAS

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1. Introduction

Let $T : A \longrightarrow B$ be a homomorphism between Banach algebras A and B. Suppose that $\overline{T(A)}$ is semi-simple. Is T necessarily continuous?

This is perhaps the most interesting open question remains in automatic continuity theory for Banach algebras. (To see [4], Open questions (16)). The continuity of homomorphisms between certain Banach algebras has been considered by several authors ([6], [9], [1], [4], [7], [3], [5], etc.). In this note we prove the following result: Let A be a complex Banach $*$–algebra with continuous involution and let B be an A^*-algebra. Let $T : A \longrightarrow B$ be an jordan $*$–homomorphism such that $\overline{T(A)} = B$. Then T is continuous (Theorem 2).

From above theorem some others results of special interest and some well-known results follow. (Corollaries 3, 4, 5, 6 and 7). We close this note with some generalizations and some remarks (Theorems 8, 9, 10 and question).

Throughout this note we consider only complex algebras. Let A and B be complex algebras. A linear mapping T from A into B is called jordan homomorphism if $T(x^2) = (Tx)^2$ for all x in A. A linear mapping $T : A \longrightarrow B$ is called spectrally-contractive mapping if $\rho(Tx) \leq \rho(x)$ for all x in A, where $\rho(x)$ denotes spectral radius of element x. Any homomorphism algebra is a spectrally-contractive mapping. If A and B are $*$–algebras, then a homomorphism $T : A \longrightarrow B$ is called $*$–homomorphism if $(Th)^* = Th$ for all self-adjoint element h in A. Recall that a Banach $*$–algebras is a complex Banach algebra with an involution $*$. An A^*-algebra A is a Banach $*$–algebra having an auxiliary norm $|\cdot|$ which satisfies B^*-condition $|x^*x| = |x|^2$ (x in A).

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Then, from [8], Lemma 4.1.14, it follows that $|Th| \leq \rho(Th)$. Therefore, we have $|Th| \leq \rho(Th) \leq \rho(h) \leq \|h\|$ for each self-adjoint element h in A. Every element x of a complex $*$−algebra A has an unique representation $x = \mu + iv$, with μ and v self-adjoint elements of A. Since the involution is continuous, it follows that any sequence (x_n) of elemints of A converges to an element x in A if only if the sequence of self-adjoint components of (x_n) converge respectively to corresponding self-adjoint components of x. Let (x_n) be a sequence of elements in A such that $x_n \to 0$ as $n \to \infty$. If $x_n = \mu_n + v_n$, where $\mu_n^* = \mu_n$ and $v_n^* = v_n, n = 1, 2, 3, \ldots$, then $|Tx_n| \leq |T\mu_n| + |Tv_n| \leq \|\mu_n\| + \|v_n\| \to 0$, as $n \to \infty$. Thus, by Closed Graph Theorem, it follows that T is continuous.

COROLLARY 3. *If T is an Jordan $*$−homomorphism from a semi-simple Banach $*$−algebra, with its range dense into an A^*-algebra, then T is continuous.*

Proof. On a semi-simple Banach algebra every involution is continuous ([6]). Apply Theorem 2.

COROLLARY 4. *If T is an Jordan $*$−homomorphism between A^*-algebras (in particular, B^*-algebras, C^*-algebras), with its range dense, then T is continuous.*

Proof. The involution in an A^*-algebra is necessarily continuous with respect to both norms ([8], Theorem 4.1.15). Apply Theorem 2.

All the above results remain valid for any $*$−homomorphism T, without density assumption on the range of T.

COROLLARY 5. *Any $*$−homomorphism from a Banach $*$−algebra with continuous involution into an A^*-algebra is continuous.*

Proof. It is well-known that the homomorphisms reduce the spectra of elements.

COROLLARY 6. *Any $*$−homomorphism from a semi-simple Banach $*$−algebra into an A^*-algebra is continuous.*

Proof. By proof of Corollary 3.

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