Bull. Korean Math. Soc. 30(1993), No. 1, pp. 117–124

QUASI O_z–SPACES

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0. INTRODUCTION

It is known that the behavior of a certain family of subsets of a completely regular space characterizes its structure. We note among others that a completely regular space X is an O_z-space (a basically disconnected space, a quasi F-space, resp.) iff every open set (cozero-set, dense cozero-set, resp.) of X is Z-embedded (C^*-embedded, resp.) (see [1], [3], and [8]).

For O_z-spaces, it is also known that a completely regular space X is an O_z-space iff υX is an O_z-space and that the real line R is an O_z-space but βR is not an O_z-space ([2] and [7]).

In this paper, we introduce a concept of quasi O_z-spaces which generalizes that of O_z-spaces. Indeed, a completely regular space X is a quasi O_z-space if for any regular closed set A in X, there is a zero-set Z in X with $A = cl_X(int_X(Z))$. We then show that X is a quasi O_z-space iff every open subset of X is $Z^\#$-embedded and that X is a quasi O_z-space iff βX is a quasi O_z-space. Furthermore, it is shown that quasi O_z-spaces are left fitting with respect to covering maps.

Observing that a quasi O_z-space is an extremally disconnected iff it is a cloz-space, the minimal extremally disconnected cover, basically disconnected cover, quasi F-cover, and cloz-cover of a quasi O_z-space X are all equivalent. Finally it is shown that a compactification Y of a quasi O_z-space X is again a quasi O_z-space iff X is $Z^\#$-embedded in Y.

For the terminology, we refer to [6].

1. QUASI O_z-SPACES

In the following, we assume that every space is a completely regular space. For a space X, let Z(X) (R(X), resp.) denote the set of all zero-sets (regular closed sets, resp.) on X.

The following are introduced by Henriksen, Vermeer, and Woods [4].

Received April 30, 1992.

NOTATION 1.1. *For any space X, let*
(a) $Z(X)^{\#} = \{ cl_X(int_X(A)) : A \in Z(X) \}$.
(b) $G(X) = \{ cl_X(C)$: *C is a cozero-set and there is a cozero-set D in X such that* $C \cap D = \emptyset$ *and* $C \cup D$ *is dense in X* $\}$

Now we introduce the concept of quasi O_z-spaces.

DEFINITION 1.2. *A space X is said to be a* ***quasi O_z-space*** *if* $Z(X)^{\#} = R(X)$.

Since a space X is an O_z-space iff $R(X) \subseteq Z(X)$, and $Z(X)^{\#} \subseteq R(X)$, every O_z-space is a quasi O_z-space. We recall that every perfectly normal space is an O_z-space and that an extremally disconnected space is an O_z-space (see [1]).

PROPOSITION 1.3. *A space X is a quasi O_z-space iff* $R(X) = Z(X)^{\#} = G(X)$.

Proof. We first note that $G(X) = \{ A \in Z(X)^{\#} : cl_X(X - A) \in Z(X)^{\#} \}$ (see [4]). Thus $G(X) \subseteq Z(X)^{\#} \subseteq R(X)$. Suppose $R(X) = G(X)$, then X is clearly a quasi O_z-space. For the converse, take any open set U in X, then $cl_X(U)$, $cl_X(X - cl_X(U)) \in R(X) = Z(X)^{\#}$; therefore $cl_X(U) \in G(X)$. Thus $R(X) = G(X)$.

DEFINTION 1.4. *Let Y be a space , then a subspace X of Y is said to be* ***$Z^{\#}$-embedded*** *in Y if for any* $A \in Z(X)^{\#}$, *there is a* $B \in Z(Y)^{\#}$ *with* $A = B \cap X$. *In case, the inclusion map* $X \hookrightarrow Y$ *is also said to be $Z^{\#}$-embedded.*

We recall that a space X is an O_z-space iff every open subset of X is Z-embedded.

THEOREM 1.5. *For any space X, the following are equivalent:*
(a) *X is a quasi O_z-space.*
(b) *Every open subset U of X is $Z^{\#}$-embedded in X.*
(c) *Every dense open subset U of X is $Z^{\#}$-embedded in X.*

Proof. (a) $\Longrightarrow$ (b) Take any $Z \in Z(U)$. Then there is a closed set A in X with $Z = A \cap U$. Since U is open, $cl_U(int_U(Z)) = cl_X(int_X(A)) \cap U$ and $cl_X(int_X(A)) \in R(X) = Z(X)^{\#}$. Thus U is $Z^{\#}$-embedded in X.
(b) $\Longrightarrow$ (c) It is trivial.

(c) $\Longrightarrow$ (a) Take any open set U in X. Put $S = U \cup (X - cl_X(U))$. Then S is open dense in X. Define a map $f : S \longrightarrow R$ by $f(x) = 0$ if $x \in U$ and $f(x) = 1$ if $x \in X - cl_X(U)$. Then f is continuous, $f^{-1}(0) = U$ and hence U is a zero-set in S. Since S is open dense in X, there is a zero-set Z in X such that $cl_S(int_S(U)) = cl_X(U) \cap S = cl_X(int_X(Z)) \cap S$. Since S is dense in X and $cl_X(U)$, $cl_X(int_X(Z))$ are regular closed sets in X, $cl_X(U) = cl_X(int_X(Z))$. Hence $R(X) \subseteq Z(X)^{\#}$, so that X is a quasi O_z-space.

LEMMA 1.6. *Let X be a space and $U \subseteq F \subseteq X$. If U is an open $Z^{\#}$-embedded subset of X, then U is $Z^{\#}$-embedded in F.*

Proof. Take any $A \in Z(U)^{\#}$. Since U is $Z^{\#}$-embedded in X, there is a $B \in Z(X)$ with $A = cl_X(int_X(B)) \cap U$. Since $B \cap F$ is closed in F and U is open in F, $cl_F(int_F(B \cap F)) \cap U = cl_U(int_U(B \cap U)) = cl_X(int_X(B)) \cap U = A$. Since $B \cap F \in Z(F)$, U is $Z^{\#}$-embedded in F.

The first half of the following is immediate from Lemma 1.6 and the second half also follows from the routine calculation.

PROPOSITION 1.7. *Let X be a quasi O_z-space and $U \subseteq X$. Then U is a quasi O_z-space if U satisfies one of the following:*
(a) U is open in X
(b) U is dense in X

Noting that for a dense subspace X of a space Y, R(X) and R(Y) are isomorphic Boolean lattices and that $Z(X)^{\#}$ is isomorphic with $Z(\beta X)^{\#}$, one has the following theorem and proposition by the above proposition.

THEOREM 1.8. *For any space X, the following are equivalent:*
(a) X is a quasi O_z-space.
(b) υX is a quasi O_z-space.
(c) βX is a quasi O_z-space.

REMARK. *The real line R is an O_z-space but βR is not an O_z-space, which is a quasi O_z-space (see [2] and [7]).*

PROPOSITION 1.9. *For any quasi O_z-space X, we have:*
(a) a $Z^{\#}$-embedded extention of X is a quasi O_z-space;
(b) every regular closed subspace of X is again a quasi O_z-space.

The following is due to Dashiell (see [3] for the detail).

DEFINITION 1.10. *A covering map $f : X \longrightarrow Y$ is said to be $Z^{\#}$-irreducible if $\{f(A) : A \in Z(X)^{\#}\} = Z(Y)^{\#}$.*

It is well known that for any covering map $f : X \to Y$, the map $\phi : R(X) \to R(Y)$ ($\phi(A) = f(A)$) is a Boolean algebra isomorphism and the inverse map ϕ^{-1} of ϕ is given by $\phi^{-1}(B) = cl_X(f^{-1}(int_Y(B)))$. Using this, one has the following.

THEOREM 1.11. *Suppose $f : X \longrightarrow Y$ is a covering map and Y is a quasi O_z-space, then f is $Z^{\#}$-irreducible and X is a quasi O_z-space.*

Proof. Since f is a covering map and Y is a quasi O_z-space, $Z(Y)^{\#} = R(Y) = \{f(A) : A \in R(X)\}$. Furthermore, for any $B \in Z(Y)^{\#}$, $cl_X(f^{-1}(int_Y(B))) \in Z(X)^{\#}$ and $B = f(cl_X(f^{-1}(int_Y(B))))$. Thus $Z(Y)^{\#} \subseteq \{f(A) : A \in Z(X)^{\#}\}$. Since $Z(Y)^{\#} \subseteq \{f(A) : A \in Z(X)^{\#}\} \subseteq \{f(A) : A \in R(X)\} = Z(Y)^{\#}$, $\{f(A) : A \in R(X)\} = \{f(A) : A \in Z(X)^{\#}\} = Z(Y)^{\#}$. So f is $Z^{\#}$-irreducible. Let $A \in R(X)$. Then there is a $B \in Z(X)^{\#}$ with $f(A) = f(B)$. Note that $A = cl_X(f^{-1}(int_Y(f(A)))) = cl_X(f^{-1}(int_Y(f(B)))) = B$. Thus $R(X) \subseteq Z(X)^{\#}$. So X is a quasi O_z-space.

DEFINITION 1.12. *A space X is said to be:*
(a) basically disconnected if every cozero-set in X is C^*-embedded in X.
(b) quasi F if every dense cozero-set in X is C^*-embedded in X.
(c) cloz if $B(X) = G(X)$.

Let $B(X)$ denote the set of clopen sets in a space X. Then it is known that a space X is basically disconnected iff $B(X) = Z(X)^{\#}$ (see [8]) and that a space X is a quasi F-space iff for any zero-sets Z_1, Z_2 with $int_X(Z_1) \cap int_X(Z_2) = \emptyset$, $cl_X(int_X(Z_1)) \cap cl_X(int_X(Z_2)) = \emptyset$. Moerover, X is extremally disconnected iff $R(X) = B(X)$.

Thus one has the following:

PROPOSITION 1.13. *For any quasi O_z-space X, the following are equivalent:*

(a) X is a cloz-space.

(b) X is a quasi F-space.

(c) X is a basically disconnected space.

(d) X is an extremally disconnected space.

2. QUASI O_z-EXTENSIONS

Let Creg denote the category of completely regular spaces and continuous maps.

DEFINITION 2.1. *Let $\underline{C}$ be a full subcategory of Creg and $X \in$ Creg.*

*(a) A pair (Y, f) is said to be a **cover** of X if $f : Y \longrightarrow X$ is a covering map.*

*(b) A pair (Y, f) is said to be a $\underline{C}$-**cover** of X if (Y, f) is a cover of X and $Y \in \underline{C}$.*

*(c) A $\underline{C}$-cover (Y, f) of X is called a **minimal** $\underline{C}$-**cover** of X if for any $\underline{C}$-cover (Z, g) of X, there is a covering map $h : Z \to Y$ with $f \circ h = g$.*

Let edc, bdc, QF, and cloz denote the full subcategories of Creg determined by extremally disconnected spaces, basically disconnected spaces, quasi-F spaces, and cloz-spaces, respectively. For any X $\in$ Creg, $(E(X), k_X)$, $(\Lambda X, \Lambda_X)$, $(QF(X), \Phi_X)$ and $(E_{cc}(X), z_X)$ denote edc-, bdc-, QF-, and cloz-cover of X, respectively (see [3], [4], [5],and [8] for the detail).

The following is immediate from Theorem 1.11 and Proposition 1.13.

THEOREM 2.2. *For any quasi O_z-space X, $(E(X), k_X)$, $(\Lambda X, \Lambda_X)$, $(QF(X), \Phi_X)$ and $(E_{cc}(X), z_X)$ are equivalent covers of X.*

PROPOSITION 2.3. *Consider the following commutative diagram :*

$$\begin{array}{ccc} P & \xrightarrow{f} & X \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_2} \\ Y & \xrightarrow{g} & W, \end{array}$$

where j_1, j_2 are dense embeddings and f, g are covering maps. Then g is $Z^\#$-irreducible and j_1 is $Z^\#$-embedded iff f is $Z^\#$-irreducible and j_2 is $Z^\#$-embedded.

Proof. ($\Longrightarrow$) Take any $A \in Z(P)^\#$, then there is a $B \in Z(Y)^\#$ with $A = B \cap P$, for j_1 is $Z^\#$-embedded. Since $f(A) = f(B \cap P) = g(B) \cap X$ and g is $Z^\#$-irreducible, $f(A) \in Z(X)^\#$. Thus f is $Z^\#$-irreducible. Let $C \in Z(X)^\#$. Then $\mathrm{cl}_P(f^{-1}(\mathrm{int}_X(C))) \in Z(P)^\#$. Since j_1 is $Z^\#$-embedded, there is a $D \in Z(Y)^\#$ such that $D \cap P = \mathrm{cl}_P(f^{-1}(\mathrm{int}_X(C)))$. Then $C = f(D \cap P) = g(D) \cap X$. Since g is $Z^\#$-irreducible $g(D) \in Z(X)^\#$; therefore j_2 is $Z^\#$-embedded.

($\Longleftarrow$) Take any $A \in Z(Y)^\#$. Then $A \cap P \in Z(P)^\#$ for P is dense in Y and $f(A \cap P) = g(A \cap P) = g(A) \cap X$; hence $g(A) \cap X \in Z(X)^\#$, because f is $Z^\#$-irreducible. Since j_2 is $Z^\#$-embedded, there is a $B \in Z(W)^\#$ with $g(A) \cap X = B \cap X$. Since j_2 is a dense embedding and $g(A)$, B are regular closed, $g(A) = B$. Thus g is $Z^\#$-irreducible.

Take any $C \in Z(P)^\#$, then $f(C) \in Z(X)^\#$. Thus there is a $D \in Z(W)^\#$ such that $f(C) = D \cap X$. Since g is a covering map, $\mathrm{cl}_Y(g^{-1}(\mathrm{int}_W(D))) \in Z(Y)^\#$. Then $f(\mathrm{cl}_Y(g^{-1}(\mathrm{int}_W(D))) \cap P) = g(\mathrm{cl}_Y(g^{-1}(\mathrm{int}_W(D)))) \cap X = D \cap X = f(C)$. Hence $\mathrm{cl}_Y(g^{-1}(\mathrm{int}_W(D))) \cap P = C$. Thus j_1 is $Z^\#$-embedded.

We recall that a space X is $Z^\#$-embedded in βX. The following theorem characterizes quasi O_z-compactifications via $Z^\#$-embedding.

THEOREM 2.4. *For any quasi O_z-space X and any compactification Y of X, the following are equivalent:*
(a) $j_1 : X \hookrightarrow Y$ is $Z^\#$ -embedded.
(b) Y is a quasi O_z-space.
(c) $E(Y) = E_{cc}(Y)$.

Proof. It is known that $\beta E(X) = E(\beta X)$ and the diagram

$$\begin{array}{ccc} E(X) & \xrightarrow{k_X} & X \\ {\scriptstyle \beta_{E(X)}}\downarrow & & \downarrow{\scriptstyle \beta} \\ E(\beta X) & \xrightarrow{k_{\beta X}} & \beta X \end{array}$$

is a pullback.

Consider the pullback diagram

$$\begin{array}{ccc} z_Y^{-1}(X) & \xrightarrow{z_{Y_X}} & X \\ {\scriptstyle j_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ E_{cc}(Y) & \xrightarrow{z_Y} & Y. \end{array}$$

Since Y is compact, there is a unique continuous map $f : \beta X \longrightarrow Y$ with $f \circ \beta_X = j_1$. It is easy to show that f is a covering map; hence there is a covering map $g : E(\beta X) \longrightarrow E_{cc}(Y)$ with $z_Y \circ g = f \circ k_{\beta X}$. Thus one has $j_1 \circ k_X = f \circ \beta_X \circ k_X = f \circ k_{\beta X} \circ \beta_{E(X)} = z_Y \circ g \circ \beta_{E(X)}$; therefore there is a unique continuous map $h : E(X) \longrightarrow z_Y^{-1}(X)$ such that $z_{Y_X} \circ h = k_X$ and $j_2 \circ h = g \circ \beta_{E(X)}$.

(a) $\Longrightarrow$ (b) It is immediate from Proposition 1.9.

(b) $\Longrightarrow$ (c) Since X is a quasi O_z-space and k_X is a covering map, k_X is $Z^{\#}$-irreducible by Theorem 1.11. Thus h and z_{Y_X} are $Z^{\#}$-irreducible. Since j_1 is dense, one has a lattice isomorphism between R(Y) and R(X) $= Z(Y)^{\#}$ via $A \mapsto A \cap X$ ($A \in R(Y)$), so that j_1 is $Z^{\#}$-embedded; therefore j_2 is also $Z^{\#}$-embedded, because of the above proposition. Thus $z_Y^{-1}(X)$ is a cloz-space; hence an extremally disconnected space and $\beta z_Y^{-1}(X) = E_{cc}(Y)$. Since h is a covering map, h is a homeomorphism. Hence $\beta E(X) = E(\beta X) = \beta z_Y^{-1}(X) = E_{cc}(Y)$. Since $E(Y) = E(\beta X)$, $E(Y) = E_{cc}(Y)$.

(c) $\Longrightarrow$ (a) Since $E_{cc}(Y)$ is an extremally disconnected space, g is a homeomorphism. Hence $z_Y \circ g = f \circ k_{\beta X}$ is $Z^{\#}$-irreducible. Since $\beta_{E(X)}$ is $Z^{\#}$-embedded, by Proposition 2.3, j_1 is $Z^{\#}$-embedded.

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