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CONTINUITY OF HOMOMORPHISMS AND DERIVATIONS ON BANACH ALGEBRAS

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1. Introduction

Let A and B be algebras. A homomorphism $\theta : A \to B$ is a linear map that satisfies

$$\theta(ab) = \theta(a)\theta(b) \quad (a, b \in A).$$

When A and B are Banach algebras, the basic automatic continuity problem is to give algebraic conditions on A and/or B which ensure that every homomorphism $\theta : A \to B$ is necessarily continuous.

In 1940 Eidelheit showed that every homomorphism of a Banach algebra onto the Banach algebra $B(X)$ of all bounded linear operators on a Banach space X is continuous. At about the same time, Gelfand proved that every homomorphism of a commutative Banach algebra into a commutative semi-simple Banach algebra is continuous. In [7] Johnson proved that every homomorphism of a Banach algebra onto non-commutative semi-simple Banach algebra is continuous, and this is still the most important result of this type.

A derivation $D : A \to A$ is a linear map such that

$$D(ab) = a\,Db + (Da)b \quad (a, b \in A).$$

In [14] Singer and Wermer showed that the range of a continuous derivation on a commutative Banach algebra is contained in the radical. In that paper they conjectured that the assumption of continuity is unnecessary. In [8] Johnson proved that every derivation on a commutative semi-simple Banach algebra is continuous and hence by the Singer-Wermer theorem it is zero.

In [15] Thomas proved the following theorem.

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THEOREM. *Any derivation on a commutative Banach algebra maps into the radical.*

In this paper we are concerned with continuity of derivations on commutative Banach algebras and of homomorphisms into commutative Banach algebras.

Throughout this paper we suppose that A is a commutative Banach algebra. R will denote the radical of A.

In § 3, we prove the following theorem.

THEOREM. *If A is an integral domain and if there exists a nonzero closed ideal I of A such that $\bigcap_{n=1}^{\infty} I^n = \{0\}$ then every derivation on A and every homomorphism of a Banach algebra onto A is continuous.*

We also prove the following theorem.

THEOREM. *Suppose that R is an integral domain. If there is a nonzero closed ideal I of R such that $\bigcap_{n=1}^{\infty} I^n = \{0\}$ then every derivation on A is continuous.*

2. Preliminaries

In this section we collect most of the concept and facts needed in the rest of the paper.

If $S : X \to Y$ is a linear map of a Banach space X into the Banach space Y, the separating space $\mathfrak{S}(S)$ or $\mathfrak{S}$ of S is defined as the set

$$\{y \in Y : \text{there are } x_n \to 0 \text{ in } X \text{ with } Sx_n \to y \text{ in } Y\}.$$

It is a closed subspace of Y. By the closed graph theorem, S is continuous if and only if $\mathfrak{S} = \{0\}$. One can find basic properties of separating spaces in [13].

The next result is called the stability lemma and due to Jewell and Sinclair [6].

THEOREM 2.1. *If $S : X \to Y$ is a linear map with separating space $\mathfrak{S}$ and $T_n : X \to X$, $R_n : Y \to Y$ are continuous linear maps such that $R_nS - ST_n$ is continuous for each $n = 1, 2, \dots$, then there is an N such that for $n \geq N$ we have*

$$(R_1 \cdots R_n \mathfrak{S})^{-} = (R_1 \cdots R_N \mathfrak{S})^{-}.$$

A closed bi-ideal J of a Banach algebra B is called a separating ideal if, for each sequence $\langle b_n \rangle$ in B, there is a natural number N such that

$$(b_1 \cdots b_n J)^{-} = (b_1 \cdots b_N J)^{-} \quad (n \geq N).$$

It is easy to show that the separating space of a homomorphism of a Banach algebra onto a Banach algebra B, or of a derivation on B, is a bi-ideal of B. From this fact and Theorem 2.1, the separating space of a homomorphism of a Banach algebra onto a Banach algebra B, or of a derivation on B, is a separating ideal of B.

A closed prime ideal P of A is called accessible, if it is the intersection of all closed ideals of A properly containing it. Otherwise, it is said to be inaccessible.

The following lemma is due to Curtis [3].

LEMMA 2.2. *If P is an accessible prime ideal of A, then P contains every separating ideal of A.*

Proof. Let I be a separating ideal of A. Suppose $I \not\subset P$. Since P is prime, $zI \neq 0$ for each $z \notin P$. Since I is a separating ideal of A, there exists $z_1, \ldots, z_{n_0}$ not in P such that for each $z \notin P$

$$(z_1 \cdots z_{n_0} z I)^{-} = (z_1 \cdots z_{n_0} I)^{-} \neq 0.$$

Let $z_0 = z_1 \cdots z_{n_0}$. Then $z_0 \notin P$. If K is a closed ideal properly containing P, pick $z \in K \setminus P$. Then $(z_0 z I)^{-} = (z_0 I)^{-} \subset K$. Thus $(z_0 I)^{-} \subset \bigcap_{K \supset P} K$, which equals P since P is accessible. However, this is impossible since neither $z_0 \in P$ nor $I \subset P$. Hence $I \subset P$.

The following theorem, known as the Mittag-Leffler theorem of Bourbaki, is essential to prove theorems in section 3. The proof can be found in [4].

THEOREM 2.3. *Let $\langle X_n : n = 0, 1, 2, \ldots \rangle$ be a sequence of complete metric spaces, and for $n = 1, 2, \ldots$, let $f_n : X_n \to X_{n-1}$ be a continuous map with $f_n(X_n)$ dense in X_{n-1}. Let $g_n = f_1 \circ \cdots \circ f_n$. Then $\bigcap_{n=1}^{\infty} g_n(X_n)$ is dense in X_0.*

3. Main Results

THEOREM 3.1. *Let A be an integral domain. If there is a nonzero closed ideal I of A such that $\bigcap_{n=1}^{\infty} I^n = \{0\}$, then $\{0\}$ is accessible.*

Proof. Note that $\{0\}$ is a prime ideal. Let H be the intersection of all nonzero closed ideals of A. Suppose H to be nonzero. Since H is closed, we have that $\overline{H^2} = H$.

For $n = 1, 2, \ldots$, let $\prod_{k=1}^{n} H$ be the product of n copies of H. Then each $\prod_{k=1}^{n} H$ is a Banach algebra. For any $n = 1, 2, \ldots$, define $f_n : \prod_{k=1}^{n+1} H \to \prod_{k=1}^{n} H$ by

$$f_n(h_1, \ldots, h_{n+1}) = (h_1, \ldots, h_{n-1}, h_n h_{n+1}),$$

where $h_i \in H$ $(i = 1, 2, \ldots, n+1)$. Then for each $n = 1, 2, \ldots$, $f_n : \prod_{k=1}^{n+1} H \to \prod_{k=1}^{n} H$ is a continuous map with $f_n(\prod_{k=1}^{n+1} H)$ dense in $\prod_{k=1}^{n} H$. Also $(f_1 \circ \cdots \circ f_n)(\prod_{k=1}^{n+1} H) = H^{n+1}$ holds.

By the Mittag-Leffler theorem, we get $\overline{\bigcap_{n=1}^{\infty} H^n} = H$. Therefore, $\bigcap_{n=1}^{\infty} H^n \neq \{0\}$. Since I is a nonzero closed ideal of A, we have that $H \subset I$ and hence $\bigcap_{n=1}^{\infty} H^n \subset \bigcap_{n=1}^{\infty} I^n = \{0\}$, which is a contradiction. Thus $\{0\}$ is accessible.

Using Lemma 2.2 and Theorem 3.1, we have the following theorem.

THEOREM 3.2. *Let A be an integral domain. If there is a nonzero closed ideal I of A such that $\bigcap_{n=1}^{\infty} I^n = \{0\}$, then every derivation on A and every homomorphism of a Banach algebra onto A is continuous.*

Proof. If there is a nonzero closed ideal I of A such that $\bigcap_{n=1}^{\infty} I^n = \{0\}$, then $\{0\}$ is accessible. By Lemma 2.2, $\{0\}$ contains every separating ideal of A. Since the separating space of a derivation on A, or of a homomorphism of a Banach algebra onto A, is a separating ideal [6], we have the result.

As shown in [12], the existence of a commutative Banach algebra which is an integral domain with inaccessible zero ideal is equivalent to the existence of a topologically simple, commutative Banach algebra other than $\mathbb{C}$. A Banach algebra A with $A^2 \neq \{0\}$ is called topologically simple if there are no closed, bi-ideals of A other than $\{0\}$ and A. Since it is not known whether there is a topologically simple, commutative Banach algebra other than $\mathbb{C}$, it is an open question whether there is a commutative Banach algebra which is an integral domain with inaccessible zero ideal.

In what follows, we consider some Banach algebras satisfying Theorem 3.2. Every derivation on each of them and every homomorphism of a Banach algebra onto each of them is continuous. These are simple consequences of Theorem 3.2.

EXAMPLE 3.3. Let A be a Banach algebra of power series. Then A is commutative and an integral domain. Note that A is a principal ideal domain. Hence for any nonzero closed ideal I of A, we have that $\bigcap_{n=1}^{\infty} I^n = \{0\}$ and every derivation on A and every homomorphism of a Banach algebra onto A is continuous.

EXAMPLE 3.4. Let ω be a weight function : ω is a continuous function on $\mathbb{R}^+$ such that $\omega(0) = 1$, $\omega(t) > 0$, $\omega(s+t) \leq \omega(s)\omega(t)$, $(s, t \in \mathbb{R}^+)$.

Let

$$L^1(\omega) = \{f : \|f\| = \int_0^{\infty} \mid f(t) \mid \omega(t)\, dt < \infty\}.$$

Then $L^1(\omega)$ is a commutative Banach algebra with the convolution multiplication:

$$(f * g)(t) = \int_0^t f(t-s)g(s)\, ds \quad (f, g \in L^1(\omega),\ t \in \mathbb{R}^+).$$

For $f \in L^1(\omega) \setminus \{0\}$, let $\alpha(f) = \inf \operatorname{supp} f$, where $\operatorname{supp} f$ is the support of f. Set $\alpha(0) = \infty$. By Titchmarsh's convolution theorem [4], $L^1(\omega)$ is an integral domain.

For $a > 0$, let

$$M_a = \{f \in L^1(\omega) : \alpha(f) \geq a\}.$$

Then M_a is a nonzero closed ideal of $L^1(\omega)$. Since $\bigcap_{n=1}^{\infty} M_a^n = \{0\}$, every derivation on $L^1(\omega)$ and every homomorphism of a Banach algebra onto $L^1(\omega)$ is continuous.

REMARK 3.5. Let D be a discontinuous derivation on a commutative Banach algebra A. Then by Theorem 2.7 of [1], there is an a in A such that $D_0 = aD$ is a discontinuous derivation on A with the following properties:

(i) $\mathcal{A}(\mathfrak{S}(D_0)) = \{x \in A : x\mathfrak{S}(D_0) = \{0\}\}$ is a closed prime ideal of A.

(ii) For every x in A, either $x \in \mathcal{A}(\mathfrak{S}(D_0))$ or $\overline{x\mathfrak{S}(D_0)} = \mathfrak{S}(D_0)$.

We now have the final result of this paper.

THEOREM 3.6. *Suppose that the radical R of A is an integral domain. If there exists a nonzero closed ideal I of R such that $\bigcap_{n=1}^{\infty} I^n = \{0\}$, then every derivation on A is continuous.*

Proof. Suppose the contrary. Then by Remark 3.5, we may assume that there is a discontinuous derivation D satisfying (i) and (ii) of Remark 3.5, so that $\mathcal{A}(\mathfrak{S})$ is a closed prime ideal of A and, for every x in A, either $x \in \mathcal{A}(\mathfrak{S})$ or $\overline{x\mathfrak{S}} = \mathfrak{S}$.

Pick any nonzero a in I. Since R is an integral domain, $a \notin \mathcal{A}(\mathfrak{S})$. Hence $\overline{a\mathfrak{S}} = \mathfrak{S}$ and $\mathfrak{S} = \overline{a\mathfrak{S}} \subset \overline{I} = I$. Thus $\bigcap_{n=1}^{\infty} \mathfrak{S}^n \subset \bigcap_{n=1}^{\infty} I^n = \{0\}$ and hence $\bigcap_{n=1}^{\infty} \mathfrak{S}^n = \{0\}$.

On the other hand, take any nonzero s in $\mathfrak{S}$. Then $s\mathfrak{S} \neq \{0\}$ implies that $\overline{s\mathfrak{S}} = \mathfrak{S}$. Since $\mathfrak{S}$ is closed, we have that $\overline{\mathfrak{S}^2} = \mathfrak{S}$. By the Mittag-Leffler theorem, we get $\overline{\bigcap_{n=1}^{\infty} \mathfrak{S}^n} = \mathfrak{S}$ and hence $\bigcap_{n=1}^{\infty} \mathfrak{S}^n$ is dense in $\mathfrak{S}$. Since D is discontinuous, $\mathfrak{S} \neq \{0\}$ and $\bigcap_{n=1}^{\infty} \mathfrak{S}^n \neq \{0\}$, which is impossible.

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