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THE BROUWER AND SCHAUDER FIXED POINT THEOREMS FOR SPACES HAVING CERTAIN CONTRACTIBLE SUBSETS

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1. Introduction

Applications of the classical Knaster-Kuratowski-Mazurkiewicz theorem [**KKM**] and the fixed point theory of multifunctions defined on convex subsets of topological vector spaces have been greatly improved by adopting the concept of convex spaces due to Lassonde [**L**]. Recently, this concept has been extended to pseudo-convex spaces, contractible spaces, or spaces having certain families of contractible subsets by Horvath [**H1–4**].

In the present paper we give a far-reaching generalization of the best approximation theorem of Ky Fan [**F1,2**] to pseudo-metric spaces and improved versions of the well-known fixed point theorems due to Brouwer [**B**] and Schauder [**S**] for spaces having certain families of contractible subsets. Our basic tool is a generalized Fan-Browder type fixed point theorem in our previous works [**P3,4**].

2. Preliminaries

A topological space X is said to be *contractible* if the identity map of X is homotopic to a constant map.

A subset C of a topological space X is said to be *compactly closed* [resp., *open*] in X if, for every compact set $K \subset X$, the set $C \cap K$ is closed [resp., open] in K.

A *convex space* X is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. See Lassonde [**L**].

Let $\langle X \rangle$ denote the set of all nonempty finite subsets of a set X.

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Let $C(X,Y)$ denote the set of all continuous functions from a topological space X into another Y.

A triple $(X, D; \Gamma)$ is called an *H-space* if X is a topological space, D a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ a family of contractible subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. If $D = X$, we denote $(X; \Gamma)$ instead of $(X, X; \Gamma)$, which is called a c-space in [**H4**].

Any convex space X is an H-space $(X; \Gamma)$ by putting $\Gamma_A = \text{co}\, A$, the convex hull of A. Other examples of $(X; \Gamma)$ are any pseudo-convex space [**H2**], any homeomorphic image of a convex space, any contractible space, and so on. See [**BC**]. Every n-simplex Δ_n is an H-space $(\Delta_n, D; \Gamma)$, where D is the set of vertices and $\Gamma_A = \text{co}\, A$ for $A \in \langle D \rangle$.

For an $(X, D; \Gamma)$, a subset C of X is said to be *H-convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$. Note that X itself and $\emptyset$ are H-convex. A subset L of X is called an *H-subspace* of $(X, D; \Gamma)$ if $L \cap D \neq \emptyset$ and for every $A \in \langle L \cap D \rangle$, $\Gamma_A \cap L$ is contractible. This is equivalent to saying that the triple $(L, L \cap D; \{\Gamma_A \cap L\})$ is an H-space.

3. Main results

We begin with the following Fan-Browder type fixed point theorem in our previous works [**P3**, Theorem 6], [**P4**, Theorem 4].

THEOREM 1. *Let $(X, D; \Gamma)$ be an H-space, Y a topological space, K a nonempty compact subset of Y, $t \in C(X, Y)$, and $S : D \to 2^Y$, $T : X \to 2^Y$ multifunctions such that*

(1) *for each $x \in D$, $Sx \subset Tx$ and Sx is compactly open;*
(2) *for each $y \in t(X)$, $T^{-1}y$ is H-convex; and*
(3) *$\overline{t(X)} \cap K \subset S(D)$.*

Suppose that either

(i) *$Y \backslash K \subset S(M)$ for some $M \in \langle D \rangle$; or*
(ii) *for each $N \in \langle D \rangle$, there exists a compact H-subspace L_N of X containing N such that $t(L_N) \backslash K \subset S(L_N \cap D)$.*

Then there exists an $x_0 \in X$ such that $tx_0 \in Tx_0$

REMARK. Theorem 1 includes Horvath [**H4**, Theorems 4.2 and 4.3].

Recall that a *gauge* $d : E \times E \to \mathbf{R}$ on a set E is a pseudo-metric (where $d(x, y) = 0$ does not necessarily imply $x = y$). A *ball* in $X \subset E$ is of the form

$$B(x, r) = \{y \in X \mid d(x, y) < r\}$$

for some $x \in X$ and $r > 0$. A function $f : X \to E$, where $X \subset E$, is said to be *d-continuous* if for each $x \in X$ and each $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(B(x, \delta)) \subset B(fx, \varepsilon)$.

From Theorem 1, we obtain the following Fan type best approximation theorem for H-spaces.

THEOREM 2. *Let $(E; \Gamma)$ be an H-space with a gauge d, X an H-subspace of E, and $f : X \to E$ a function such that*

(1) *every ball in E is H-convex; and*
(2) *f is d-continuous.*

Suppose that there exists a nonempty compact subset K of X such that either

(i) *there exists an $M \in \langle X \rangle$ such that for each $x \in X \backslash K$, $d(fx, y) < d(fx, x)$ for some $y \in M$; or*
(ii) *for each $N \in \langle X \rangle$, there exists a compact H-subspace L_N of X containing N such that for each $x \in L_N \backslash K$, $d(fx, y) < d(fx, x)$ for some $y \in L_N$.*

Then there exists an $x_0 \in K$ such that

$$d(fx_0, x_0) \leq d(fx_0, y) \quad \textit{for all} \quad y \in X.$$

Proof. Suppose that for each $y \in K$, we have

$$d(fy, y) > d(fy, X) = \inf\{d(fy, x) \mid x \in X\}.$$

Define $S : X \to 2^X$ by

$$Sx = \{y \in X \mid d(fy, x) < d(fy, y)\}$$

for $x \in X$. We show that S satisfies all of the requirements of Theorem 1 with $X = D = Y$, $S = T$, and $t = 1_X$.

(1) In order to show Sx is open for each $x \in X$, let $y \in Sx$. For the $\varepsilon > 0$ satisfying $d(fy, x) = d(fy, y) - \varepsilon$, we have a $\delta > 0$ such that f maps $B(y, \delta)$ into $B(fy, \varepsilon/4)$. Let $\delta_1 = \min\{\delta, \varepsilon/4\}$ and $y' \in B(y, \delta_1)$. Then

$$\begin{aligned} d(fy', x) &\leq d(fy', fy) + d(fy, x) < \frac{\varepsilon}{4} + d(fy, y) - \varepsilon \\ &\leq d(fy, fy') + d(fy', y') + d(y', y) - \frac{3\varepsilon}{4} \\ &\leq \frac{\varepsilon}{4} + d(fy', y') + \frac{\varepsilon}{4} - \frac{3\varepsilon}{4} = d(fy', y') - \frac{\varepsilon}{4} \\ &< d(fy', y'), \end{aligned}$$

because $d(fy, fy') < \varepsilon/4$ and $d(y', y) < \varepsilon/4$. Therefore, $d(fy', x) < d(fy', y')$ and hence $y' \in Sx$. Consequently, for any $y \in Sx$, there exists a $\delta_1 > 0$ such that $B(y, \delta_1) \subset Sx$, whence Sx is open by (1).

(2) For each $y \in X$, $S^{-1}y = X \cap B(fy, d(fy, y))$ is H-convex since it is the intersection of two H-convex subsets.

(3) Clearly $S^{-1}y \neq \emptyset$ for each $y \in K$.

Further, (i) and (ii) imply (i) and (ii) of Theorem 1, resp. Therefore, by Theorem 1, there exists an $\overline{x} \in X$ such that $\overline{x} \in S\overline{x}$, that is, $d(f\overline{x}, \overline{x}) < d(f\overline{x}, \overline{x})$, a contradiction. This completes our proof.

REMARK. If $X = K$ is compact in Theorem 2, then the "coercivity" conditions (i) and (ii) of Theorem 2 are satisfied automatically. For this case, the origin of Theorem 2 goes back to Cellina [**C**] for metric locally convex spaces and to Fan [**F1**] for normed vector spaces, both in 1969. Later, in 1977, Rassias [**R**] obtained Theorem 2 for a compact convex subset X of a metric topological vector space E where every ball is convex.

From Theorem 2, we have the following Brouwer type fixed point theorem for H-spaces.

THEOREM 3. *Let $(X; \Gamma)$ be a compact metric H-space such that every ball is H-convex. Then every continuous function $f : X \to X$ has a fixed point.*

Proof. Put $y = fx_0$ in the conclusion of Theorem 2.

REMARK. Clearly, Theorem 3 generalizes the well-known results of Brouwer [**B**] and Schauder [**S**, Satz I]. Theorem 3 was noted by Rassias [**R**] for a compact convex subset of a metric topological vector space and by Park [**P1**, Corollary 13.2] for a metric compact convex space.

Note that if X is a convex space, then by putting $L_N = \mathrm{co}(M \cup N)$, (i) implies (ii) in Theorem 2. Note also that if a topological vector space E has a seminorm, then every ball is convex. In fact, from Theorem 2, we have the following by the method in [**P2**].

THEOREM 4. *Let E be a seminormed vector space, X a convex subset of E, and $f : X \to E$ a continuous function. Suppose that there exist a nonempty compact subset K of X and, for each $N \in \langle X \rangle$, a compact convex subset L_N of X containing N such that $x \in L_N \setminus K$ implies*

$$\|fx - y\| < \|fx - x\| \quad \text{for some} \quad y \in L_N.$$

Then there exists an $x_0 \in K$ such that

$$\|fx_0 - x_0\| \le \|fx_0 - y\| \quad \text{for all} \quad y \in W(x_0).$$

In Theorem 4, $W(x_0)$ is the closure of one of the following sets:

$$I_X(x_0) = \{x_0 + r(u - x_0) \in E \mid u \in X,\ r > 0\},$$
$$O_X(x_0) = \{x_0 + r(u - x_0) \in E \mid u \in X,\ r < 0\}.$$

Theorem 4 improves the main result of [**P2**] and extends many well-known results including Ky Fan [**F2**, Theorem 7].

As another application of Theorem 1, we have the following fixed point theorem.

THEOREM 5. *Let $(X, D; \Gamma)$ be an H-space whose topology has a Hausdorff uniform structure, K a nonempty compact subset of X, and $t \in \mathcal{C}(X, K)$. Suppose that, for each entourage V, there exist two multifunctions $S : D \to 2^X$ and $T : X \to 2^X$ satisfying (1)–(3) of Theorem 1 and Graph$(T) \subset V$. Then t has a fixed point.*

Proof. Note that $t(X) \subset K$ implies Condition (ii) of Theorem 1. Let V be any entourage of the uniform structure. Then, by Theorem

1, there exist a multifunction $T : X \to 2^X$ and an $x_0 \in X$ such that

$$(x_0, tx_0) \in \text{Graph}(T) \subset V.$$

Therefore, for any entourage V, t has a V-fixed point. Since $\overline{t(X)} \subset K$ is compact, t must have a fixed point.

REMARK. Note that if $X = D$, then Theorem 4 reduces to Horvath [**H4**, Theorem 4.4].

From Theorem 5, we have the following Schauder type fixed point theorem for H-spaces.

THEOREM 6. *Let $(X, D; \Gamma)$ be a metric H-space such that every ball is H-convex. Then every compact continuous function $f : X \to X$ has a fixed point.*

REMARK. Theorem 6 includes Theorem 3 and Schauder [**S**, Satz II].

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