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A CERTAIN SUBGROUP OF THE FUNDAMENTAL GROUP OF A TRANSFORMATION GROUP

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1. Introduction

F.Rhodes [2] introduced the fundamental group $\sigma(X, x_0, G)$ of a transformation group (X, G) as an extension of the fundamental group of a topological space X. B.J.Jiang [1] introduced the Jiang subgroup $J(f, x_0)$ of the fundamental group $\pi_1(X, f(x_0))$ of a topological space X and a self-map f of X and showed that $J(f, x_0)$ is contained in $Z(f_\pi(\pi_1(X, x_0)), \pi_1(X, f(x_0)))$. M.H.Woo and S.H.Han [3] introduced the extended Jiang subgroup $J(f, x_0, G)$ of the fundamental group of a transformation group (X, G). But it is unknown that $J(f, x_0, G)$ is contained in $Z(f_\sigma(\sigma(X, x_0, G))$, $\sigma(X, f(x_0), G))$.

In this paper, we want to find a subgroup $HJ(f, x_0, G)$ of the extended Jiang subgroup of a transformation group which is contained in $Z(f_\sigma(\sigma(X, x_0, G)), \sigma(X, f(x_0), G))$ and is an extension of the Jiang subgroup $J(f, x_0)$. That is, if the acting group G is the trivial group $\{1_X\}$, then this is the Jiang's result.

2. Definitions and Results

Let (X, G, π) be a transformation group, where X is a path connected space with x_0 as base point. Given an element g of G, *a path α of order g* with base point x_0 is a continuous map $\alpha : I \longrightarrow X$ such that $\alpha(0) = x_0$ and $\alpha(1) = gx_0$. A path α_1 of order g_1 and a path α_2 of order g_2 give rise to a path $\alpha_1 + g_1\alpha_2$ of order $g_1 g_2$ defined by the equations

$$(\alpha_1 + g_1\alpha_2)(s) = \begin{cases} \alpha_1(2s), & 0 \le s \le 1/2 \\ g_1\alpha_2(2s-1), & 1/2 \le s \le 1. \end{cases}$$

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Two paths α and α' of the same order g are said to be *homotopic* if there is a continuous map $F : I^2 \longrightarrow X$ such that

$$F(s,0) = \alpha(s) \qquad 0 \le s \le 1,$$
$$F(s,1) = \alpha'(s) \qquad 0 \le s \le 1,$$
$$F(0,t) = x_0 \qquad 0 \le t \le 1,$$
$$F(1,t) = gx_0 \qquad 0 \le t \le 1.$$

The homotopy class of a path α of order g was denoted by $[\alpha; g]$. Two homotopy classes of paths of different orders g_1 and g_2 are distinct, even if $g_1x_0 = g_2x_0$. F. Rhodes showed that the set of homotopy classes of paths of prescribed order with the rule of composition $\circ$ form a group, where $\circ$ is defined by $[\alpha_1; g_1] \circ [\alpha_2; g_2] = [\alpha_1 + g_1\alpha_2; g_1g_2]$. This group was denoted by $\sigma(X, x_0, G)$, and was called the *fundamental group* of (X, G) with base point x_0.

Let f be a selfmap of X. A homotopy $H : X \times I \longrightarrow X$ is said to be an *f-cyclic homotopy* if $H(\cdot, 0) = f = H(\cdot, 1)$. In this case, the path $H(x_0, \cdot)$ is called a trace of the homotopy H. In [2], Jiang has defined $J(f, x_0) = \{[h] \in \pi_1(X, f(x_0)) | h$ is homotopic to a trace of an f-cyclic homotopy $\}$. For a transformation group (X, G), a homotopy $H : X \times I \longrightarrow X$ is said to be an *f-cyclic homotopy of order g* if $H(\cdot, 0) = f, H(\cdot, 1) = gf$. In [3], Woo and Han has defined $J(f, x_0, G) = \{[\alpha; g] \in \sigma(X, f(x_0), G) | \alpha$ is homotopic to a trace of an f-cyclic homotopy of order $g\}$. In this paper, the acting group G is abelian.

DEFINITION 1. $HJ(f, x_0, G) = \{[\alpha; g] \in \sigma(X, f(x_0), G) | \alpha$ is homotopic to a trace of an f-cyclic homotopy H of order g such that $H(g'x_0, \cdot) = g'H(x_0, \cdot)$ for each $g' \in G\}$.

THEOREM 1. *Let (X, G) be a transformation group and f be an endomorphism of (X, G). Then $HJ(f, x_0, G)$ is a subgroup of $\sigma(X, f(x_0), G)$.*

Proof. Let $[\alpha_1; g_1]$ and $[\alpha_2; g_2]$ be any elements of $HJ(f, x_0, G)$. Then there exist f-cyclic homotopies K, K' of order g_1, g_2 with trace α_1, α_2 respectively such that $K(g'x_0, \cdot) = g'K(x_0, \cdot)$ and $K'(g'x_0, \cdot) = g'K'(x_0, \cdot)$ for each $g' \in G$. Define a homotopy $F : X \times I \longrightarrow X$ by

$$F(x,t) = \begin{cases} K(x, 2t), & 0 \le t \le 1/2 \\ g_1K'(x, 2t-1), & 1/2 \le t \le 1. \end{cases}$$

This is well defined and continuous. Now we have $F(x,0) = f(x)$ and $F(x,1) = g_1K'(x,1) = g_1g_2f(x)$. It is easy to show that the trace of F is $\alpha_1 + g_1\alpha_2$ and $F(g'x_0,\cdot) = g'F(x_0,\cdot)$ for each $g' \in G$. This means that $[\alpha_1; g_1] \circ [\alpha_2; g_2]$ belongs to $HJ(f, x_0, G)$. Next, we show that if $[\alpha; g] \in HJ(f, x_0, G)$, then $[\alpha; g]^{-1} \in HJ(f, x_0, G)$. Let $[\alpha; g] \in HJ(f, x_0, G)$. Then there exists an f-cyclic homotopy K of order g with trace α such that $K(g'x_0,\cdot) = g'K(x_0,\cdot)$. Define $F(x,t) = g^{-1}K(x, 1-t)$. Then F is an f-cyclic homotopy of order g^{-1} with trace $g^{-1}\alpha\rho$ such that $F(g'x_0,\cdot) = g'F(x_0,\cdot)$,where $\rho(t) = 1 - t$. Thus $[\alpha; g]^{-1}$ belongs to $HJ(f, x_0, G)$.

If we take the acting group $G = \{1_X\}$, then we obtain $HJ(f, x_0, G) = J(f, x_0)$. From this fact, $HJ(f, x_0, G)$ is an extension of the concept of Jiang subgroup $J(f, x_0)$. Let f be a homomorphism from (X, G) to (X', G). Then f induces a homomorphism $f_\sigma : \sigma(X, x_0, G) \longrightarrow \sigma(X', f(x_0), G)$ given by $f_\sigma([\alpha; g]) = [f\alpha; g]$.

THEOREM 2. *If $f : (X, G) \longrightarrow (X, G)$ is an endomorphism , then $HJ(f, x_0, G)$ is contained in $Z(f_\sigma(\sigma(X, x_0, G)), \sigma(X, f(x_0), G))$.*

Proof. Let $[\alpha; g]$ be any element of $HJ(f, x_0, G)$. Then there exists an f-cyclic homotopy $K : X \times I \longrightarrow X$ of order g with trace α such that $K(g'x_0,\cdot) = g'K(x_0,\cdot)$ for each $g' \in G$. For each $[\beta; g'] \in \sigma(X, x_0, G)$, we must show that $[\alpha; g] \circ (f_\sigma([\beta; g'])) = (f_\sigma([\beta; g'])) \circ [\alpha; g]$. That is, $\alpha + gf\beta$ is homotopic to $f\beta + g'\alpha$. Let $J : I \times I \longrightarrow X$ be given by $J = K(\beta \times 1)$. Define $F : I \times I \longrightarrow X$ by

$$F(s,t) = \begin{cases} J(2s(1-t), 2st), & 0 \le s \le 1/2 \\ J(1-(2-2s)t, (2-2s)t + 2s - 1), & 1/2 \le s \le 1. \end{cases}$$

Then

$$F(s,0) = \begin{cases} J(2s, 0), & 0 \le s \le 1/2 \\ J(1, 2s-1), & 1/2 \le s \le 1 \end{cases}$$

$$= \begin{cases} K(\beta(2s), 0), & 0 \le s \le 1/2 \\ K(g'x_0, 2s-1), & 1/2 \le s \le 1 \end{cases}$$

$$= (f\beta + g'\alpha)(s),$$

$$F(s,1) = \begin{cases} J(0,2s), & 0 \le s \le 1/2 \\ J(2s-1,1), & 1/2 \le s \le 1 \end{cases}$$
$$= \begin{cases} K(x_0,2s), & 0 \le s \le 1/2 \\ K(\beta(2s-1),1), & 1/2 \le s \le 1 \end{cases}$$
$$= (\alpha + gf\beta)(s),$$

$F(0,t) = J(0,0) = K(x_0,0) = f(x_0)$ and $F(1,t) = J(1,1) = K(gx_0,1) = g(f(g'x_0)) = gg'f(x_0)$. Therefore we obtain $[\alpha + gf\beta; gg'] = [f\beta + g'\alpha; gg']$.

Let $\alpha : I \longrightarrow X$ be a path such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Then α induces an isomorphism $\alpha_* : \sigma(X,x_0,G) \longrightarrow \sigma(X,x_1,G)$ such that $\alpha_*([\beta;g]) = [\alpha\rho + \beta + g\alpha; g]$.[2]

THEOREM 3. *Let $F : X \times I \longrightarrow X$ be a homotopy from f to f' such that $F(g'x_0,\cdot) = g'F(x_0,\cdot)$, then $HJ(f,x_0,G)$ and $HJ(f',x_0,G)$ are isomorphic.*

proof. Let $F : X \times I \to X$ be a homotopy between f and f' such that $F(g'x_0,\cdot) = g'F(x_0,\cdot)$. Let $p(t) = F(x_0,t)$ for all $t \in I$. Then p is a path from $f(x_0)$ to $f'(x_0)$. Since $p_* : \sigma(X,f(x_0),G) \to \sigma(X,f'(x_0),G)$ is an isomorphism, it is sufficient to show that

$$p_*(HJ(f,x_0,G)) \subset HJ(f',x_0,G).$$

Let $[\alpha; g]$ be any element of $HJ(f,x_0,G)$. Then there exists an f-cyclic homotopy $G : X \times I \longrightarrow X$ of order g with trace α such that $G(\cdot,0) = f, G(\cdot,1) = gf$ and $G(g'x_0,\cdot) = g'G(x_0,\cdot)$ for each $g' \in G$. Consider a homotopy $K : X \times I \longrightarrow X$ given by

$$K(x,t) = \begin{cases} F(x,1-3t), & 0 \le t \le 1/3 \\ G(x,3t-1), & 1/3 \le t \le 2/3 \\ gF(x,3t-2), & 2/3 \le t \le 1. \end{cases}$$

Then $K(x,0) = F(x,1) = f'(x), K(x,1) = gF(x,1) = gf'(x)$,

$$K(x_0,t) = \begin{cases} F(x_0,1-3t), & 0 \le t \le 1/3 \\ G(x_0,3t-1), & 1/3 \le t \le 2/3 \\ gF(x_0,3t-2), & 2/3 \le t \le 1 \end{cases}$$

$$= (p\rho + \alpha + gp)(t) = p_*(\alpha)(t),$$

and $K(g'x_0, \cdot) = g'K(x_0, \cdot)$ for each $g' \in G$. Thus $p_*([\alpha; g])$ belongs to $HJ(f', x_0, G)$.

COROLLARY 4. *If two selfmaps f, f' of X are homotopic, then $J(f, x_0)$ is isomorphic to $J(f', x_0)$.*

THEOREM 5. *If $f : (X, G) \longrightarrow (X, G)$ is an endomorphism and x_1 belongs to the orbit of x_0, then $HJ(f, x_0, G)$ is isomorphic to $HJ(f, x_1, G)$.*

Proof. Let p be a path in X from x_0 to x_1. Then fp is a path from $f(x_0)$ to $f(x_1)$. Since $(fp)_* : \sigma(X, f(x_0), G) \longrightarrow \sigma(X, f(x_1), G)$ is an isomorphism, it is sufficient to show $(fp)_*(HJ(f, x_0, G)) \subset HJ(f, x_1, G)$. Let $[\alpha; g]$ be any element of $HJ(f, x_0, G)$. Then there is an f-cyclic homotopy $K : X \times I \longrightarrow X$ of order g with trace α such that $K(g'x_0, \cdot) = g'K(x_0, \cdot)$. Since $(fp)_*([\alpha; g]) = [fp\rho + \alpha + gfp; g] = [\beta; g]$ and $K(g'x_1, \cdot) = K(g'hx_0, \cdot) = g'hK(x_0, \cdot) = g'K(x_1, \cdot)$, where $\beta = K(x_1, \cdot), x_1 = hx_0$ for some $h \in G$, therefore $(fp)_*([\alpha; g])$ belongs to $HJ(f, x_1, G)$.

DEFINITION 2. A transformation group (X, G) is called a transformation H-group with base point x_0 if there exists a map $\mu : X \times X \longrightarrow X$ such that $\mu(gx, y) = \mu(x, gy) = g\mu(x, y)$ and $\mu(x, x_0) = x = \mu(x_0, x)$.

THEOREM 6. *If (X, G) is a transformation H-group with base point $f(x_0)$, then we have $HJ(f, x_0, G) = \sigma(X, f(x_0), G)$.*

Proof. Let $[\alpha; g]$ be any element of $\sigma(X, f(x_0), G)$ and μ be the transformation H-group structure. Define $K : X \times I \longrightarrow X$ by $K = \mu(f \times \alpha)$. Then $K(x, 0) = \mu(f(x), \alpha(0)) = \mu(f(x), f(x_0)) = f(x)$, $K(x, 1) = \mu(f(x), gf(x_0)) = gf(x)$, $K(x_0, t) = \mu(f(x_0), \alpha(t)) = \alpha(t)$ and $K(g'x_0, \cdot) = g'K(x_0, \cdot)$ for each $g' \in G$. Thus $[\alpha; g]$ belongs to $HJ(f, x_0, G)$.

EXAMPLE. Let $X = G = S^1$ and $\pi : X \times G \longrightarrow X$ be given by $\pi(e^{i\theta_1}, e^{i\theta_2}) = e^{i(\theta_1 + \theta_2)}$. Then (X, G, π) is a transformation group. Moreover, we can take a transformation H-group structure $\mu : X \times X \longrightarrow X$ given by $\mu(e^{i\theta_1}, e^{i\theta_2}) = e^{i(\theta_1 + \theta_2)}$ with the base point 1. Thus we have from Theorem 6 that $HJ(f, 1, S^1) = \sigma(S^1, f(1), S^1)$.

Let (X, G, π) be a transformation group and X^X be the function space with the compact open topology. Define $H(X^X) = \{f \in X^X \mid$ for any $g \in G, fg = gf$ on the base point $x_0\}$ with the subspace topology of X^X. Then $(H(X^X), G, \pi')$ is a transformation group where $\pi'(f, g) = gf$. Let $p : X^X \longrightarrow X$ be the evaluation map given by $p(f) = f(x_0)$. Then $p : (H(X^X), G) \longrightarrow (X, G)$ is a homomorphism and induces a homomorphism $p_\sigma : \sigma(H(X^X), f, G) \longrightarrow \sigma(X, f(x_0), G)$ given by $p_\sigma([\alpha; g]) = [p\alpha; g]$.

THEOREM 7. *Let X be a CW-complex. Then $p_\sigma(\sigma(H(X^X), f, G)) = HJ(f, x_0, G)$.*

Proof. Let $[\alpha; g]$ be any element of $\sigma(H(X^X), f, G)$. Then $\alpha : I \longrightarrow H(X^X)$ is a path of order g with base point f. Since $\alpha : I \longrightarrow H(X^X) \subset X^X, \alpha \equiv i \circ \alpha : I \longrightarrow X^X$ is a continuous map and hence α is a path from f to gf, where $i : H(X^X) \to X^X$ is the inclusion map. Then $\phi(\alpha) : X \times I \longrightarrow X$ given by $\phi(\alpha)(x, t) = \alpha(t)(x)$ satisfies that $\phi(\alpha)(x, 0) = \alpha(0)(x) = f(x), \phi(\alpha)(x, 1) = \alpha(1)(x) = gf(x), \phi(\alpha)(x_0, t) = \alpha(t)(x_0) = p\alpha(t)$ and $\phi(\alpha)(g'x_0, t) = \alpha(t)(g'x_0) = g'\alpha(t)(x_0) = g'\phi(\alpha)(x_0, t)$. Thus $\phi(\alpha) : X \times I \longrightarrow X$ is an f-cyclic homotopy of order g with trace $p\alpha$ such that $\phi(\alpha)(g'x_0, t) = g'\phi(\alpha)(x_0, t)$ for each $g' \in G$. Therefore, $p_\sigma[\alpha; g] = [p\alpha; g]$ belongs to $HJ(f, x_0, G)$.

Conversely, let$[\alpha; g]$ be any element of $HJ(f, x_0, G)$. Then there exists an f-cyclic homotopy $F : X \times I \longrightarrow X$ of order g with trace α and $F(g'x_0, t) = g'F(x_0, t)$. Since we can define $\bar{F} : I \longrightarrow X^X$ by $\bar{F}(t) = F(\cdot, t), \bar{F}(t)(g'x_0) = F(g'x_0, t) = g'F(x_0, t) = g'\bar{F}(t)(x_0)$ for any $g' \in G$. Thus $\bar{F}(t)$ belongs to $H(X^X)$ and hence $[\bar{F}; g]$ belongs to $\sigma(H(X^X), f, G)$. Since $p\bar{F}(s) = \bar{F}(s)(x_0) = F(x_0, s) = \alpha(s)$. Therefore $[\alpha; g]$ belongs to $p_\sigma(\sigma(H(X^X), f, G))$.

THEOREM 8. *Let f, k be endomorphisms of (X, G).*

(1) *$HJ(k, f(x_0), G) \subset HJ(kf, x_0, G)$,*

(2) *$k_\sigma(HJ(f, x_0, G)) \subset HJ(kf, x_0, G)$, where $k_\sigma[\alpha; g] = [k\alpha; g]$.*

Proof. (1) Let $[\alpha; g]$ be an element of $HJ(k, f(x_0), G)$. Then there exists a k-cyclic homotopy $K : X \times I \longrightarrow X$ of order g with trace α such that $K(g'f(x_0), \cdot) = g'K(f(x_0), \cdot)$. If we define a homotopy $K' : X \times I \longrightarrow X$ by $K' = K(f \times 1)$, then K' is a kf-cyclic homotopy of order g with trace α such that $K'(g'x_0, \cdot) = g'K(x_0, \cdot)$. Thus $[\alpha; g]$ belongs to $HJ(kf, x_0, G)$.

(2) Let $[\alpha; g]$ be an element of $HJ(f, x_0, G)$ and $k_\sigma : \sigma(X, f(x_0), G) \longrightarrow \sigma(X, kf(x_0), G)$ be a homomorphism. Then there exists an f-cyclic homotopy $K : X \times I \longrightarrow X$ of order g with trace α such that $K(g'x_0, \cdot) = g'K(x_0, \cdot)$. If we define $K' : X \times I \longrightarrow X$ by $K' = kK$, then $K'(x, 0) = kK(x, 0) = kf(x), K'(x, 1) = kK(x, 1) = kgf(x) = gkf(x), K'(x_0, t) = kK(x_0, t) = k\alpha(t)$ and $K'(g'x_0, \cdot) = g'K'(x_0, \cdot)$. Therefore, $k_\sigma([\alpha; g]) \in HJ(kf, x_0, G)$.

REMARK. If we take $G = \{1_X\}$, (2) implies the Jiang's result,that is, $k_\pi(J(f, x_0)) \subset J(kf, x_0)$.

THEOREM 9. *If* $f, k : (X, G) \longrightarrow (X, G)$ *are isomorphisms and* $f(x_0) = k(x_0)$, *then* $HJ(f, x_0, G)$ *is equal to* $HJ(k, x_0, G)$.

Proof. Let $[\alpha; g]$ be any element of $HJ(f, x_0, G)$. Then there exists an f-cyclic homotopy $K : X \times I \longrightarrow X$ of order g with trace α such that $K(g'x_0, \cdot) = g'K(x_0, \cdot)$. Define $K' : X \times I \longrightarrow X$ by $K' = K(f^{-1}k \times 1)$. Then K' is a k-cyclic homotopy of order g with trace α such that $K'(g'x_0, \cdot) = g'K'(x_0, \cdot)$. Thus $[\alpha; g]$ belongs to $HJ(k, x_0, G)$. Similarly, $HJ(k, x_0, G)$ is contained in $HJ(f, x_0, G)$. Therefore $HJ(f, x_0, G)$ is equal to $HJ(k, x_0, G)$

References

1. B.J.Jiang, ***Lectures on Nielsen fixed point theory***, Contemp. Math. **14** (1983).
2. F.Rhodes, ***On the fundamental group of a transformation group***, Proc. London Math. Soc. **16** (1966), 635-650.
3. M.H. Woo and S.H. Han, ***An extended Jiang subgroup of the fundamental group of a transformation group***, Comm. of Korean Math. Soc. **28** (1991), 135-143.

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