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THE EXTENSION OF SOLUTIONS FOR THE CAUCHY PROBLEM IN THE COMPLEX DOMAIN II

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1. Introduction

J. Leray [7] proposed a sufficient condition for the solvability of the Cauchy problem on the initial hyperplane $x_1 = 0$ with Cauchy data which are holomorphic with respect to the variables parallel to some analytic subvariety S of the initial hyperplane. Limiting the problem to the case of operators with constant coefficients, A. Kaneko [2] proposed a new sharper sufficient condition. Later we generalized this condition and showed that it is necessary and sufficient for the solvability of the Cauchy problem for the hyperfunction Cauchy data and the distribution Cauchy data which contain variables parallel to S as holomorphic parameters in [5, 6].

In this paper, we extend the results in [6] to the case of operators with variable coefficients and show that it is sufficient for the solvability of the Cauchy problem for the hyperfunction Cauchy data. Our main theorem can be considered as an example of a deep theorem on micro-hyperbolic systems by Kashiwara-Schapira [4] and we give a direct proof based on an elementary sweeping out procedure developed in Kaneko [3].

Let $P(x, D)$ be an m-th order linear partial differential operator with real analytic coefficients, and let $P_m(x, D)$ be its principal part. Assume that the initial plane $x_1 = 0$ is non-characteristic with respect to P. We use the following notations for separation of the independent variables; $x = (x_1, x') = (x_1, x'', x''')$ with $x'' = (x_2, \ldots, x_{k+1})$ and similar notation for the complexification $z = x + \sqrt{-1}\, y$, and also for the dual variables $\zeta = \xi + \sqrt{-1}\, \eta$. Now, let Γ be a convex open cone in $\mathbf{R}^k$

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and Δ a convex open cone such that $\Delta \subset\subset \Gamma$, i.e., $\overline{\Delta} \cap S^{n-1}$ is a compact subset of $\Gamma \cap S^{n-1}$. Also, suppose that $\{B_j(x, D)\}_{j=0}^{m-1}$ is a normal system of boundary operators. In other words, for each j $B_j(x, D)$ is a j-th order differential operator with real analytic coefficients and the initial plane $x_1 = 0$ is non-characteristic with respect to B_j.

2. Main Results

Let us consider the following holomorphic Cauchy problem

$$(1) \qquad \begin{cases} P(z, D)u(z) = 0 \\ B_j(z, D)u\big|_{z_1=0} = u_j(z'), \quad j = 0, \ldots, m-1 \end{cases}$$

where holomorphic data $u_j(z')$ are given on a domain of the form $\{z' \in \mathbf{C}^{n-1} \mid |x'| < A_1,\ y'' \in \Gamma,\ |y'| < B_1\}$.

LEMMA. *Assume that if*

$$|Re\, z_1| \le B_1,\ |Re\, z'| \le B_2,\ |Im\, z| \le B_3,\ Re\, \zeta'' \in \Delta^\circ,$$

then for some $a, b > 0$, the solutions $\zeta_1 = \tau_j(z, \zeta')$, $j = 1, \ldots, m$, of $P_m(z, \zeta) = 0$ satisfy

$$(2) \qquad -Im\, \tau_j(z, \zeta') \le a(|y''|\,|Re\, \zeta''| + |Im\, \zeta''|) + b|\zeta'''|.$$

Then the solution of (1) can be continued onto the domain

$$W = \{z \in \mathbf{C}^n \mid 0 \le x_1 < T,\ |x'| < A,\ y'' \in \Delta,\ \frac{1}{r^2}|y_1| < \delta c^2(1 - \sqrt{x_1})^2|y''|, \delta|y''| < B\}$$

where A, B, r, c, δ are suitable positive constants.

Proof. It follows from the Cauchy-Kowalevsky theorem that the solution exists on a domain

$$\tilde{W} = \{z \in \mathbf{C}^n \mid |x'| < A_1/2,\ |z_1| < r^2c^2\text{dis}(y'', \partial\Gamma),\ |y''| < B_1/2,\ y'' \in \Delta\}$$

If $y'' \in \Delta$, then there exists $\delta > 0$ such that

$$\delta \leq \mathrm{dis}(y'', \partial\Gamma), \quad |y''| = 1.$$

Thus, for every $y'' \in \Delta$,

$$\delta |y''| \leq \mathrm{dis}(y'', \partial\Gamma) \tag{3}$$

Let $\varepsilon > 0$ be a small parameter and consider the following real hypersurface in $\mathbf{C}^n$

$$\frac{1}{r^2}\,|y_1| = c^2(1-\sqrt{x_1})^2(\delta|y''| - (1+kx_1)\varepsilon)^2 \tag{4}$$

Then we claim that there exist constants $k > 0$ and $T > 0$ independent of ε so that the hypersurface (4) has no characteristic normal vector on $0 < x_1 \leq T$. We employ $-\mathrm{Re}\langle z, \sqrt{-1}\,\zeta\rangle$ as the real Euclidean structure of $\mathbf{C}^n$. Then the normal vector at a point $x + \sqrt{-1}\,y$ is

$$\begin{aligned}
\xi &= (\frac{\gamma}{r^2}, -2c^2\delta(1-\sqrt{x_1})^2(\delta|y''| - (1+kx_1)\varepsilon)\frac{1}{|y''|}y'', 0, \cdots, 0) \\
\eta &= (2k\varepsilon c^2(1-\sqrt{x_1})^2(\delta|y''| - (1+kx_1)\varepsilon) \\
&\quad + \frac{c^2}{\sqrt{x_1}}(1-\sqrt{x_1})(\delta|y''| - (1+kx_1)\varepsilon)^2, 0\ldots 0)
\end{aligned}$$

where $-1 \leq \gamma \leq 1$. If $y'' \in \Delta$, We have obviously, $\xi'' \cdot y'' < 0$. Hence $\xi'' \in -\Delta^\circ$. Therefore, the vector $(-\xi, -\eta)$ must satisfy (2) if the point is characteristic. The condition (2) implies that $\eta_1 \leq a|y''||\xi''|$. On the other hand, if $|y''| < 4\varepsilon/\delta$,

$$\begin{aligned}
\eta_1 &\geq 2k\varepsilon c^2(1-\sqrt{x_1})^2(\delta|y''| - (1+kx_1)\varepsilon) \\
&> \frac{k}{4}|y''||\xi''| > a|y''||\xi''|
\end{aligned}$$

as long as we choose $k > 4a$. If $|y''| \geq 4\varepsilon/\delta$, we have, similarly,

$$\begin{aligned}
\eta_1 &\geq \frac{c^2}{\sqrt{x_1}}(1-\sqrt{x_1})(\delta|y''| - (1+kx_1)\varepsilon)^2 \\
&\geq \frac{1}{2\sqrt{x_1}}(\frac{3-kx_1}{4})|y''||\xi''| > a|y''||\xi''|
\end{aligned}$$

as long as we choose $T > 0$ such that

$$\sqrt{T} < \frac{-4a + \sqrt{16a^2 + 3k}}{k}.$$

Thus, for a sufficiently large k and a corresponding small T, we have verified our claim. Now put

$$\Omega_\varepsilon = \{0 \leq x_1 < T, |x'| < A', \frac{1}{r^2}|y_1| < c^2(1 - x_1)^2(\delta|y''| - (1 + kx_1)\varepsilon)^2,$$
$$y'' \in \Delta,\ (1 + kx_1)\varepsilon < \delta|y''| < B'\}.$$

If T and B' are sufficiently small, by (3), the point on the boundary $x_1 = 0$ or $\delta|y''| = B'$ or $\delta|y''| = (1 + kx_1)\varepsilon$ is contained in $\tilde{W}$ where the solution $u(z)$ is already defined. Similarly, by the Cauchy-Kowalevsky theorem, we can assume that $u(z)$ is defined on $|x'| = A'$, because the initial data are real analytic on $|x'| = A'$. On the point $0 < x_1 < T$, $(1 + kx_1)\varepsilon < \delta|y''| < B'$, the boundary has no characteristic normal vector as was shown above. Finally, on the point $x_1 = T$, the normal vectors are the nonnegative linear combination of $\xi = 0$, $\eta = (1, 0, \ldots, 0)$ and those vectors considered above. Thus applying a refinement of Zerner's theorem in [1] we can conclude that $u(z)$ can be continued to the open set $\Omega_0 = W$.

THEOREM. *Assume that if* $|Re\, z_1| \leq A_1$, $|Re\, z'| \subseteq A_2$, $|Im\, z| \leq A_3$, $Re\,\zeta'' \in \Delta^\circ$, *then for some* $a, b > 0$, *the solutions* $\zeta_1 = \tau_j(z, \zeta')$, $j = 1, \ldots, m$ *of* $P_m(z, \zeta) = 0$ *satisfy*

$$-Im\,\tau_j(z, \zeta') \leq a(|y''|\,|Re\,\zeta''| + |Im\,\zeta''|) + b|\zeta'''|.$$

Assume that the hyperfunction data $u_j(x')$, $j = 0, \ldots, m - 1$ *can be expressed as the boundary values of functions* $F_j(z')$ *holomorphic in* $\{\mathbf{R}^{n-1} \times \sqrt{-1}(\Gamma \times \mathbf{R}^{n-k-1})\} \cap \{|z'| < \alpha\}$. *Then in a neighborhood of the origin, we can solve the following boundary value problem*

$$\begin{cases} P(x, D)u = 0 \\ B_j(x, D)u\big|_{x_1 \to +0} = u_j(x'), \quad j = 0, \ldots, m - 1 \end{cases}$$

Proof. With the initial data $F_j(z')$, we are going to solve the holomorphic Cauchy problem (1). Put $A_1 = B_1 = \alpha/\sqrt{2}$. Then by Lemma, the holomorphic solution $F(z)$ can be continued to W which is a wedge with its edge tangent to the real axis. Thus $F(z)$ continued there defines a hyperfunction solution $u(x)$ of $P(x, D)u = 0$ on $x_1 > 0$ locally on a neighborhood of the origin. Moreover, the boundary values of u agree with the given data u_j [2]. Therefore the proof is complete.

Similarly, we can prove the following condition is sufficient for the solvability of the boundary value problem on sufficient condition on $x_1 < 0$.

$$\begin{gathered}\operatorname{Im} \tau_j(z, \zeta') \le a(|y''||\operatorname{Re} \zeta''| + |\operatorname{Im} \zeta''|) + b|\zeta'''|, \\ \text{if } P_m(z, \zeta) = 0, \ \operatorname{Re} \zeta'' \in \Delta^\circ,\end{gathered}$$

for some constants $a, b > 0$. Therefore, we obtain

COROLLARY. *Assume that if* $|Re\, z_1| \le A_1$, $|Re\, z'| \le A_2$, $|Im\, z| \le A_3$, $Re\zeta'' \in \Delta^\circ$, *then for some* $a, b > 0$, *the solutions* $\zeta_1 = \tau_j(z, \zeta')$, $j = 1, \ldots, m$, *of* $P_m(z, \zeta) = 0$ *satisfy*

$$|Im\, \tau_j(z, \zeta')| \le a(|y''||Re\, \zeta''| + |Im\, \zeta''|) + b|\zeta'''|.$$

Assume that the hyperfunction data $u_j(x')$, $j = 0, \ldots, m-1$ *can be expressed as the boundary values of functions* $F_j(z')$ *holomorphic in* $\{\mathbf{R}^{n-1} \times \sqrt{-1}(\Gamma \times \mathbf{R}^{n-k-1})\} \cap \{|z'| < \alpha\}$. *Then the Cauchy problem*

$$\begin{cases} P(x, D)u = 0 \\ B_j(x, D)u\big|_{x_1=0} = u_j(x'), \quad j = 0, \ldots, m-1 \end{cases}$$

admits a hyperfunction solution which contains the same holomorphic parameters.

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