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STABILITY OF CLOSED SUBGROUPS $H(k,t)$ IN $SU(3)$

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1. Preliminaries and main theorem.

1.1. We review Mashimo-Tasaki's work which gives a necessary and sufficient condition for a torus H in a compact connected semisimple Lie group G with a biinvariant Riemannian metric to be stable (cf.[3]).

Let T be a maximal torus of G. We denote by g (resp. t) the Lie algebra of G (resp. T). Let g^c be the complexification of g. We denote by Δ the set of all nonzero roots of g^c with respect to t^c. Let B be the killing form of g^c. We define an inner product $<,>$ on g by $< X,Y >:= -B(X,Y), (X,Y \in g)$. The problem with respect to the stability is independent of the choice of biinvariant metric. Here and throughout this paper, we use the biinvariant Riemannian metric on G which is induced by $<,>$. For $\alpha \in \Delta, X_\alpha$ being defined by $B(X,X_\alpha) = \alpha(X)$ for any $X \in t^c$ belongs to it. Here and from now on, we put $i := \sqrt{-1}$. We put

$$\Gamma := \{X \in t | \exp(X) = \epsilon\}.$$

Let h be the Lie algebra of a given a torus H. We put

$$D(H) := \{X \in h | < \mu, X > \in 2\pi Z \text{ for any } \mu \in h \cap \Gamma\}.$$

Let $P : t \to h$ be the orthogonal projection. Mashimo-Tasaki [3, Lemma 2.1] obtained the following:

LEMMA A. *Let G be a compact connected semisimple Lie group and H be a torus of G. Then H is unstable if and only if there exist $\lambda, \mu \in D(H)$ and $\beta \in \Delta$ such that $(|\lambda|^2 + |\mu|^2) < |P(i\beta)|^2$ with $\lambda + \mu = P(\beta)$.*

1.2. We consider $G = SU(3)$ and take as a torus H of $SU(3)$, $H(k,t) := \{diag(e^{ik\theta}, e^{it\theta}, e^{-i(k+t)\theta}) | \theta \in R\}$, $|k| + |t| \neq 0 (k,t \in Z)$.

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Here $diag(x, y, z)$ denotes a 3×3 diagonal matrix whose diagonal entries are x, y and z. Then we have the following main Theorem:

Theorem. *$H(k,t), |k| + |t| \neq 0 (k, t \in Z)$, in $SU(3)$ with biinvariant Riemannian metric are unstable.*

2. Proof of the main theorem.

2.1. The Lie algebra $sl(3,C)$ of $SL(3,C)$ is the complexification of the real Lie algebra $su(3)$. Here $su(3)$ denotes the Lie algebra of $SU(3)$.

Let t be a subspace of $su(3)$ which consists of the diagonal matrices of trace 0. Then the complexification t^c of t is a Cartan subalgebra of $sl(3,C)$. We define a linear map $e_j : t^c \to C, 1 \leq j \leq 3$, by $e_j(H)$:=(the (j,j)-entry of H) for $H \in t^c$. Then the non-zero roots of $sl(3,C)$ with respect to t^c are

$$e_i - e_j, \quad (1 \leq i, j \leq 3, \quad i \neq j). \tag{1}$$

We put

$$\alpha := e_1 - e_2, \quad \beta := e_2 - e_3, \quad \gamma := e_1 - e_3. \tag{2}$$

The killing form B of $sl(3,C)$ satisfies

$$B(X,Y) = 6 \quad Trace(XY), (X, Y \in sl(3,C)). \tag{3}$$

Then

$$\begin{cases} \alpha = X_\alpha = diag(\frac{1}{6}, \ \frac{-1}{6}, \ 0), \\ \beta = X_\beta = diag(0, \ \frac{1}{6}, \ \frac{-1}{6}), \\ \gamma = X_\gamma = diag(\frac{1}{6}, \ 0, \ \frac{-1}{6}). \end{cases} \tag{4}$$

We define the canonical inner product $<,>$ on $su(3)$ by

$$< X, Y > := -B(X,Y) = -6 \quad Trace(XY), \ (X, Y \in su(3)), \tag{5}$$

and define the biinvariant Riemannian metric g on $SU(3)$ which is induced by $<,>$. Γ in t becomes

$$(6) \qquad \Gamma = \{diag(2\pi im, 2\pi in, -2\pi i(m+n)) | m,n \in Z\}.$$

Let $h(k,t)$ be the Lie algebra of $H(k,t)$, $|k|+|t| \neq 0 (k,t \in Z)$. Then

$$(7) \qquad h(k,t) = \{diag(ik\theta, it\theta, -i(k+t)\theta) | \theta \in R\}.$$

2.2. In case of $(2t+k) \neq 0$.

I) $kt > 0$: In this case, we have an orthogonal decomposition. $t = h(k,t) + h^{\perp}(k,t)$ with respect to $<,>$. Then

$$(8) \qquad \begin{cases} h(k,t) = \{diag(ik\theta, it\theta, -i(k+t)\theta) | \theta \in R\}, \\ h^{\perp}(k,t) = \{diag(i\theta, -\frac{2k+t}{2t+k} i\theta, \frac{k-t}{2t+k} i\theta) | \theta \in R\}. \end{cases}$$

We obtain from (6) and (8)

$$(9) \qquad h(k,t) \cap \Gamma = \begin{array}{c} \{diag(2\pi im, 2\pi itm/k, -2\pi im(k+t)/k) | \\ m(\in Z) \text{ satisfies the condition } (tm)/k \in Z\}. \end{array}$$

We get from (5),(8) and (9)

$$(10) \qquad D(H(k,t)) = \{diag(\frac{ik}{c}\frac{k\theta}{m_0}, \frac{it}{c}\frac{k\theta}{m_0}, \frac{-i(k+t)}{c}\frac{k\theta}{m_0}) | \theta \in Z\},$$

where $c := 12(k^2+kt+t^2)$, and $m_0 := min\{|m| | tm/k \in Z, m \in (Z-0)\}$. From now on, we use c and m_0 in this paper. We have from (4) and (8)

$$(11) \qquad P(i\beta) = (k+2t) \cdot diag(ik/c,\ it/c,\ -i(k+t)/c).$$

We put $a_0 := tm_0/k \in Z$. Then $t = a_0 k/m_0$. Now we take two elements λ, μ of $D(H(k,t))$ as follows.

$$(12) \qquad \begin{cases} \lambda = k \cdot diag(ik/c,\ it/c,\ -i(k+t)/c), \\ \mu = 2t \cdot diag(ik/c,\ it/c,\ -i(k+t)/c). \end{cases}$$

Then we get from (5),(11) and (12)

$$(13)\qquad \begin{cases} |\lambda|^2 = (12k^4 + 12k^3t + 12k^2t^2)/c^2 = k^2/c, \\ |\mu|^2 = (48k^2t^2 + 24kt^3 + 48t^4)/c^2, \\ |P(i\beta)|^2 = (12k^4 + 60k^3t + 108k^2t^2 + 96kt^3 + 48t^4)/c^2, \\ (\lambda + \mu) = P(i\beta). \end{cases}$$

Hence, if $kt > 0, |P(i\beta)|^2 - (|\lambda|^2 + |\mu|^2) > 0$. Thus the main theorem in this case is obtained from Lemma A.

II) $kt < 0$: (8),(9) and (10) are valid in this case. We have from (4) and (8)

$$(14)\qquad P(i\alpha) = (k - t)\cdot diag(ik/c,\ it/c,\ -i(k+t)/c).$$

We choose two elements as follows.

$$(15)\qquad \begin{cases} \lambda = k \cdot diag(ik/c,\ it/c,\ -i(k+t)/c), \\ \mu = -t \cdot diag(ik/c,\ it/c,\ -i(k+t)/c). \end{cases}$$

Then we obtain from (5),(14) and (15)

$$(16)\qquad \begin{cases} |\lambda|^2 = k^2/c,\ |\mu|^2 = t^2/c, \\ |P(i\alpha)|^2 = (k-t)^2/c,\ (\lambda + \mu) = P(i\alpha). \end{cases}$$

Moreover, if $kt < 0, |P(i\alpha)|^2 - (|\lambda|^2 + |\mu|^2) > 0$. Thus, using Lemma A, we get the main Theorem in this case.

III) $k \neq 0$ and $t = 0$: In this case, $h(k,0)$ and $h^{\perp}(k,0)$ in the orthogonal decomposition of t are as follows.

$$(17)\qquad \begin{cases} h(k,0) = \{diag(ik\theta,\ 0,\ -ik\theta)|\ \theta \in R\}, \\ h^{\perp}(k,0) = \{diag(i\theta,\ -2i\theta,\ i\theta)|\ \theta \in R\}. \end{cases}$$

We have from (6) and (17)

$$(18)\qquad h(k,0) \cap \Gamma = \{diag(2\pi im,\ 0,\ -2\pi im)|\ m \in Z\}.$$

We obtain from (5),(17) and (18)

$$(19) \qquad D(H(k,0)) = \{diag(i\theta,\ 0,\ -i\theta)|\ \theta \in \frac{1}{12}Z\}.$$

We have from (4) and (17)

$$(20) \qquad P(i\gamma) = diag(i/6,\ 0,\ -i/6).$$

We take an element λ of $D(H(k,0))$ as follows.

$$(21) \qquad \lambda = diag(i/12,\ 0,\ -i/12).$$

Then we get from (5),(20) and (21)

$$(22) \qquad |\lambda|^2 = 1/12,\ |P(i\gamma)|^2 = 1/3,\ 2\lambda = P(i\gamma).$$

Hence, $|P(i\gamma)|^2 - 2|\lambda|^2 = 1/6 > 0$. Thus, using Lemma A, we obtain the main Theorem in this case.

IV) $k = 0$ and $t \neq 0$: Similarly, we get from (7)

$$(23) \qquad \begin{cases} h(0,t) = \{diag(0,\ it\theta,\ -it\theta)|\ \theta \in R\}, \\ h^{\perp}(0,t) = \{diag(2i\theta,\ -i\theta,\ -i\theta)|\ \theta \in R\}. \end{cases}$$

We get from (6)and(23)

$$(24) \qquad h(0,t) \cap \Gamma = \{diag(0,\ 2\pi im,\ -2\pi im)|\ m \in Z\}.$$

We obtain from (5),(23) and (24)

$$(25) \qquad D(H(0,t)) = \{diag(0,\ i\theta,\ -i\theta)|\ \theta \in \frac{1}{12}Z\}.$$

We have from (4) and (23)

$$(26) \qquad P(i\beta) = diag(0,\ i/6,\ -i/6).$$

We choose an element of λ of $D(H(0,t))$ as follows.

$$\lambda = diag(0,\ i/12,\ -i/12). \tag{27}$$

Then we obtain from (5), (26) and (27)

$$|\lambda|^2 = 1/12,\ |P(i\beta)|^2 = 1/3,\ 2\lambda = P(i\beta). \tag{28}$$

Thus, using Lemma A, we get the main Theorem in this case. Therefore, in case of $(2t+k) \neq 0$, the proof of the main Theorem is completed.

2.3. In case of $2t + k = 0$.

$h(2k,-k)$ and $h^{\perp}(2k,-k)$ in the orthogonal decomposition of t are as follows.

$$\begin{cases} h(2k,-k) = \{diag(2ik\theta,\ -ik\theta,\ -ik\theta) \mid \theta \in R\}, \\ h^{\perp}(2k,-k) = \{diag(0,\ i\theta,\ -i\theta) \mid \theta \in R\}. \end{cases} \tag{29}$$

We get from (6) and (29)

$$h(2k,-k) \cap \Gamma = \{diag(-4\pi im,\ 2\pi im,\ 2\pi im) \mid \theta m \in Z\}. \tag{30}$$

We obtain from (5),(29) and (30)

$$D(H(2k,-k)) = \{diag(i\theta/18,\ -i\theta/36,\ -i\theta/36) \mid \theta \in Z\}. \tag{31}$$

We have from (4)and (29)

$$P(i\alpha) = diag(i/6,\ -i/12,\ -i/12). \tag{32}$$

We take two elements $\lambda, \mu \in D(H(2k,-k))$ as follows.

$$\begin{cases} \lambda = diag(i/18,\ -i/36,\ -i/36), \\ \mu = diag(i/9,\ -i/18,\ -i/18). \end{cases} \tag{33}$$

Then we get from (5), (32) and (33)

$$|P(i\alpha)|^2 = 1/4,\ |\lambda|^2 = 1/36,\ |\mu|^2 = 1/9,\ (\lambda + \mu) = P(i\alpha). \tag{34}$$

Hence, using Lemma A, we get the main Theorem in the case of $(2t + k) = 0$.

2.4. We get the main Theorem from 2.2 and 2.3.

References

[1] Joon-Sik Park, Harmonic inner automorphisms of compact connected semisimple Lie groups, Tohoku Math. J. 42(1990), 83-91

[2] M.Kreck and S.Stolz, Some homomorphic but not diffeomorphic homogeneous 7-manifolds with positive sectional curvature, to appear.

[3] K.Mashimo and H. Tasaki, Stability of maximal tori in compact Lie groups, to appear.

[4] M.Sugiura, Unitary representations and Harmonic Analysis, North - holland, 1990.

[5] H.Urakawa, Stability of harmornic maps and eigenvalues of the Laplacian, Trans. Amer. Math. Soc. 301(1987), 557-589.

[6] N. Wallach, Harmonic Analysis on Homogeneous Spaces, Marcel Dekker, New York, 1973.

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