

APPROXIMATE FIBRATIONS AND EXAMPLES

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1. Introduction.

Approximate fibrations were introduced by D. Coram and P. Duvall in [6] as a generalization of both Hurewicz fibrations and cell-like maps. Much of the theory of Hurewicz fibrations carries over to the set of approximate fibrations.

A proper map $p : M \mapsto B$ between locally compact ANRs is called an **approximate fibration** if it has the following homotopy property: Given an open cover ϵ of B , an arbitrary space X and two maps $g : X \mapsto M$ and $F : X \times I \mapsto B$ such that $p \circ g = F_0$, there exists a map $G : X \times I \mapsto M$ such that $G_0 = g$ and $p \circ G$ is ϵ -close to F .

In [9], L.S. Husch gave an example of a closed manifold M and an approximate fibration $p : M^n \mapsto S^1 (n \geq 6)$ such that p can not be approximated by a Hurewicz fibration. T.A. Chapman and S. Ferry [5] gave an example of an approximate fibration $p : M^n \mapsto S^2 (n \geq 5)$ which is homotopic to a locally trivial bundle map, but can not be approximated by a locally trivial bundle map.

Let F^m be a compact manifold. We denote by $S(F)$ the set of equivalence classes of the form $[f]$, where $f : M^m \mapsto F^m$ is a homotopy equivalence of a compact manifold M^m to F^m which is a homeomorphism of ∂M to ∂F . Another such map $f' : M' \mapsto F$ is defined to be equivalent to f provided that there exists a homeomorphism $h : M \mapsto M'$ for which $h \circ f' \simeq f$. If T^n is the n -torus and $e : T^n \mapsto T^n$ is any standard finite cover, then there is a transfer map $\hat{e} : S(T^n \times F) \mapsto S(T^n \times F)$ defined by

$$\hat{e}([f]) = [\tilde{f}],$$

where \tilde{f} comes from the pull-back diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & T^n \times F \\ \downarrow & & \downarrow e \times Id \\ M & \xrightarrow{f} & T^n \times F \end{array}$$

We use $S_0(T^n \times F)$ to denote those elements of $S(T^n \times F)$ that are invariant under any of these transfer maps.

The following notations will be used through this paper. If α is an open cover of Y , then a homotopy $h_t : X \mapsto Y$ is an α -homotopy provided that each set $\{h_t(X) | 0 \leq t \leq 1\}$ lies in some element of α .

A proper map $f : X \mapsto Y$ is said to be α -equivalence if there is a proper map $g : Y \mapsto X$ and proper homotopies $\phi_t : g \circ f \simeq Id_X$, $\theta_t : f \circ g \simeq Id_Y$ such that $f \circ \phi_t : X \mapsto Y$ and $\theta_t : Y \mapsto Y$ are α -homotopies. We write this as

$$\phi_t : g \circ f \simeq Id$$

and

$$\theta_t : f \circ g \simeq Id$$

where $f^{-1}(\alpha)$ denotes the open cover of X defined by

$$f^{-1}(\alpha) = \{f^{-1}(U) | U \in \alpha\}.$$

If α is an open cover, then a proper map $f : X \mapsto Y$ is said to be an α -fibration if for all maps $F : Z \times [0, 1] \mapsto Y$, and $\tilde{F}_0 : Z \mapsto X$ for which $f \circ \tilde{F}_0 = F_0$, there is a map $G : Z \times [0, 1] \mapsto X$ such that $G_0 = \tilde{F}_0$ and $f \circ G$ is α -close to F . This latter statement means that given any $(z, t) \in Z \times [0, 1]$, there is an element U of α containing both $f \circ G(z, t)$ and $F(z, t)$. Note that a proper map $f : X \mapsto Y$ is an approximate fibration provided that it is an α -fibration for every open cover α of Y .

In this paper, we give conditions that approximate fibrations over Euclidean space R^n can be approximated by locally trivial bundle maps, and give examples of approximate fibrations which can not be approximated by locally trivial bundle maps.

2. Approximate fibrations over Euclidean space.

In this section, we give the necessary and sufficient condition that approximate fibrations over R^n can be approximated by trivial bundle maps.

In [4], T.A. Chapman gave the following nice results by applying controlled engulfing and torus geometry. These are the key results to determine whether approximate fibrations can be approximated by locally trivial bundle maps.

THEOREM 2.1. *Let $n \geq 0$ be an integer. For any $\epsilon > 0$, there exists a $\delta > 0$ so that $f : M^{m+n} \mapsto R^n \times F^m$ is a $p^{-1}(\delta)$ -equivalence for which $f|_{\partial M} : \partial M \mapsto R^n \times \partial F^m$ is a homeomorphism, where M^{m+n} is a manifold, F^m is a compact manifold with boundary and $m+n \geq 5$, then there is an element $\sigma(f)$ of $S_0(T^n \times F)$ which vanishes if and only if f is $p^{-1}(\epsilon)$ -homotopic to a homeomorphism (p denotes the projection onto R^n).*

As an immediate consequence of Theorem 2.1, we have the following.

THEOREM 2.2. *Let $p : M^{m+n} \mapsto R^n$ be an approximate fibration whose fiber is homotopy equivalent to a closed manifold F , and $m+n \geq 5$. Then for some $\pi^{-1}(\epsilon)$ -equivalence $f : M \mapsto R^n \times F$, $\sigma(f)$ vanishes if and only if p is approximated by a trivial bundle (π denotes the projection to R^n)*

Proof. "Only if": It is obvious by Theorem 2.1.

"If": Suppose p is approximated by a trivial bundle $q : M \mapsto R^n$. Then there is a fiber preserving homeomorphism $h : M \mapsto R^n \times F$. Since p and q are arbitrarily close, $h : M \mapsto R^n \times F$ is a $\pi^{-1}(\epsilon)$ -equivalence with $\sigma(h) = 0$ for arbitrary small $\epsilon > 0$.

The following corollary shows that any approximate fibration over R^n with some special fiber is unique up to isomorphism.

COROLLARY 2.3. *Let $p : M^{m+n} \mapsto R^n$ and $q : \tilde{M}^{m+n} \mapsto R^n$ be approximate fibrations whose fibers are homotopy equivalent to a closed manifold F^m satisfying $S_0(T^n \times F) = 0$, and $m+n \geq 5$. Then for any $\epsilon > 0$, there is a homeomorphism $h : M \mapsto \tilde{M}$ such that p and $q \circ h$ are ϵ -close.*

proof. It is obvious by Theorem 2.2.

COROLLARY 3.3. *Let $p : M^{m+n} \mapsto R^n$ be an approximate fibration whose fiber is homotopy equivalent to a closed manifold which is a $K(\pi, 1)$ with π poly Z , and $m + n \geq 5$. Then p is approximated by a trivial bundle.*

Proof. We obtain that $S(T^n \times F^m) = 0$ from [4]. Then the result follows from Theorem 2.2.

COROLLARY 3.4. *Let $p : M^{m+n} \mapsto R^n$ be an approximate fibration whose fiber is homotopy equivalent to S^n , and $m + n \geq 5$. Then p is approximated by a trivial bundle.*

Proof. Since $S(T^n \times S^n) = 0$ from [4], the result follows from Theorem 2.2.

3. Examples.

In this section, we give examples of approximate fibrations which can not be approximated by locally trivial bundles.

Before we proceed further, we give a result [4], which is used in Examples 3.2 and 3.3.

PROPOSITION 3.1. *Let B be a space which is locally polyhedral, α an open cover of B , and $m \geq 5$ be an integer. Then there exists an open cover β of B so that if M^m is a manifold ($\partial M = \emptyset$) and $f : M \mapsto B$ is a β -fibration, then f is α -close to an approximate fibration $p : M \mapsto B$.*

In the following examples, we show that some approximate fibrations $p : M^{m+n} \mapsto T^n$ and $p : M^{m+n} \mapsto R^n$ whose fibers are homotopy equivalent to $N^m = L^3 \times T^{m-3}$ can not be approximated by locally trivial bundle maps for $m + n \geq 6$, where L^3 is a Lens space with $\pi_1(L) \cong Z_{p^2}$ (p : prime) and T^n is the n -dimensional torus.

Example 3.2. Let L^3 be a 3-dimensional Lens space with $\pi_1(L) \cong Z_{p^2}$ (p : prime), and T^{m-2} be the $(m-2)$ -dimensional torus. We consider $M_0^{m+1} = L^3 \times T^{m-2}$ ($m+1 \geq 6$). Rewrite $M_0 = L^3 \times T^{m-2}$ as $N^m \times S^1$, where $N = L^3 \times T^{m-3}$.

Farrel and Hsiang [8] showed that there exists an h -cobordism $(W^{n+2}; M_0^{m+1}, M_1^{m+1})$ such that M_0 is not homeomorphic to M_1 , but the obstruction to splitting M_1 into $N' \times R$ vanishes, where N' is a codimension 1 submanifold of M_1 . We consider a homotopy equivalence $f_1 : M_1 \mapsto M_0$, which is the composition of the following maps

$$M_1 \xrightarrow{\text{inclusion}} W \xrightarrow{\text{deformation retract}} M_0$$

Let $\pi : M_0 \mapsto S^1$ be the projection onto the second factor. We consider the following pull-back diagrams

$$\begin{array}{ccc} M & \xrightarrow{f} & N \times S^1 \\ \downarrow & & \downarrow Id \times \nu \\ M_1 & \xrightarrow{f_1} & N \times S^1 \end{array}$$

and

$$\begin{array}{ccc} \bar{M} & \xrightarrow{f} & N \times R \\ \downarrow & & \downarrow Id \times e \\ M_1 & \xrightarrow{f_1} & N \times S^1, \end{array}$$

where $\nu : S^1 \mapsto S^1$ is defined by $\nu(z) = z^k$ (k : positive integer) and $e : S^1 \mapsto S^1$ by $e(x) = e^{\pi i x}$.

Since the obstruction to splitting M_1 into $N' \times R$ vanishes, we can restore f_1 by the wrapping up process (see [4,10]), and hence there exists a homeomorphism $h : M \mapsto M_1$ such that $f \circ h \simeq f_1$ (see Lemma 7.3,[4]).

If k is large enough, $\pi \circ f : M \mapsto S^1$ is close to an approximate fibration $p : M \mapsto S^1$ by Proposition 3.1, where $\pi : N \times S^1 \mapsto S^1$ is the projection.

claim: $p : M \mapsto S^1$ can not be approximated by a locally trivial bundle.

Otherwise, p is approximated by a locally trivial bundle $q : M \mapsto S^1$. Then q is ϵ -homotopic to $\pi \circ f$ for arbitrary small $\epsilon > 0$. By the homotopy lifting property, there exists a fiber homotopy equivalence $g : M \mapsto N \times S^1$ such that $f \simeq g$.

Since $\chi(S^1) = 0$, the Whitehead torsion $\tau(g)$ in $Wh(N \times S^1)$ is zero [1], and hence

$$\tau(f_1) = \tau(f) = 0.$$

Then the h-cobordism $(W; M_0, M_1)$ is homotopic to the product. Thus, $M_0 \cong M_1$. This is a contradiction. Now consider $M_0 = L^3 \times T^{m-2} =$

$N \times S^1 \times S^1$, where $N = L^3 \times T^{m-4}$. Form the following pull-back

$$\begin{array}{ccc}
 M & \xrightarrow{f} & N \times S^1 \times S^1 \\
 \downarrow & & \downarrow Id \times \nu \times Id \\
 \bar{M} & \xrightarrow{\bar{f}} & N \times S^1 \times S^1 \\
 \downarrow & & \downarrow Id \times Id \times \nu \\
 M_1 & \xrightarrow{f_1} & N \times S^1 \times S^1
 \end{array}$$

By the same argument, we conclude that there is a homeomorphism $h : M \rightarrow M_1$ such that $f_1 \circ h \simeq f$. Hence, for large enough k , $\pi \circ f : M \rightarrow S^1 \times S^1$ is close to an approximate fibration $p : M \rightarrow S^1 \times S^1$, but p can not be approximated by a locally trivial bundle, where $\pi : N \times S^1 \times S^1 \rightarrow S^1 \times S^1$ is the projection.

Similarly we can extend so that there are some approximate fibrations $p : M^{m+n} \rightarrow T^n$ ($m+n \geq 6$) whose fibers are homotopy equivalent to $N^M = L^3 \times T^{m-3}$ which can not be approximated by locally trivial bundle maps.

Example 3.3. In the previous example, there exists an h-cobordism $(W; M_0^{m+1}, M_1^{m+1})$ such that M_0 is not homeomorphic to M_1 , but the obstruction to splitting M_1 into $N' \times R$ vanishes for $M_0 = L^3 \times T^{m-2}$, where N' is a codimension 1 submanifold of M_1 . Without loss of generality we assume a natural homotopy equivalence $f_1 : M_1 \rightarrow M_0 = N \times S^1$ as a $\pi^{-1}(\epsilon)$ -equivalence for small enough $\epsilon > 0$.

Consider the pull-back diagram

$$\begin{array}{ccc}
 M & \xrightarrow{f} & N \times R^1 \\
 \downarrow & & \downarrow Id \times e \\
 M_1 & \xrightarrow{f_1} & N \times S^1,
 \end{array}$$

where $e : R^1 \rightarrow S^1$ is the covering projection defined by $e(x) = e^{\pi i x}$ for $x \in R^1$. Then f is a $q^{-1}(\delta)$ -equivalence, δ depends on the size of ϵ , and $q : N \times R^1 \rightarrow R^1$ is the projection onto the 2-nd factor.

For such a f , $q \circ f : M \mapsto R^1$ is a 2δ -fibration, and hence $q \circ f$ is close to an approximate fibration $p : M \mapsto R^1$ by Proposition 3.1.

claim $p : M \mapsto R^1$ can not be approximated by a trivial bundle.

Otherwise, there exists a homeomorphism $h : M \mapsto N \times R^1$ such that h is $\pi^{-1}(\epsilon)$ -close to f . Since the obstruction to splitting M_1 is zero, by the wrapping up process, we can recover $f_1 : M_1 \mapsto N \times S^1$. In other words, there is a $q^{-1}(\epsilon)$ -equivalence $\hat{f} : M^1 \mapsto N \times S^1$ such that $\hat{f} \simeq f_1$. By the uniqueness of the wrapping up process [4], we find a homeomorphism $\hat{h} : M_1 \mapsto N \times S^1 = M_0$. This is a contradiction.

By the similar inductive procedure in Example 3.2, we can extend so that there are some approximate fibrations $p : M^{m+n} \mapsto R^n$ ($m+n \geq 6$) whose fiber is homotopy equivalent to $N^m = L^3 \times T^{m-3}$ which can not be approximated by trivial bundle maps.

References

1. D R. Anderson, *The Whitehead torsion of a fiber homotopy equivalence*, Michigan Math. J. **21** (1974), 171-180
2. T A. Chapman, *Lectures on Hilbert cube manifolds*, C B M.S. Regional Conf Series in Math. **28**, 1976.
3. T.A. Chapman, *Approximating maps into fiber bundles by homeomorphism*, Rocky Mountain J. Math. **10** (1980), 333-350.
4. T.A. Chapman, *Approximation results in topological manifolds*, Memoirs of Amer. Math. Soc. **34** (1981)
5. T.A. Chapman and S. Ferry, *Constructing approximate fibrations*, Trans. Amer. Math. Soc. **276** (1983), 757-774
6. D. Coram and P. Duvall, *Approximate fibrations*, Rocky Mountain J. Math. **7** (1977), 255-288
7. F.T. Farrell, *The obstruction to fibering a manifold over the circle*, Bull. Amer. Math. Soc. **73** (1967), 737-740
8. F.T. Farrell and W.C. Hsiang, *H-cobordant manifolds are not necessarily homeomorphic*, Bull. Amer. Math. Soc. **73** (1967), 741-744.
9. L S. Husch, *Approximating approximate fibrations by fibrations*, Canad. J. Math. **29** (1977), 897-913
10. L.C. Siebenmann, *A total Whitehead torsion obstruction to fibering over circle*, Comm. Math. Helvetici **45** (1970), 1-48
11. E H. Spanier, *Algebraic topology*, McGraw Hill, New York, 1966.

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