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A STUDY ON MULTIOBJECTIVE FRACTIONAL OPTIMIZATION PROBLEMS

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1. Introduction

Duality theorems for single objective fraction optimization problems have been of much intrest in the past [1,4,8,10,11,12]. Recently there has been of growing intrest in studying optimality theorems and duality theurems for multyobjective fractional optimazation problems [2,3,5,13,14,15]. In particular optimazition problem in which numerators of objective functions involves the square roots of quadratic forms and established optimality theorems and duality theorems in the framework of the Hanson-Mond classes of functions. Also Bhatia and Jain [1] entended Singh's results to a nondifferentiable multiobjective fractional optimization problem in which numerators of objective functions involves the square roots of quadratic forms.

In this paper, we consider the following nondifferentiable multiobjective optimization problem (p):

$$(P) Minimize\ F(X) = \left(\frac{f_1(x) + (xD_1x)^{\frac{1}{2}}}{h_1(x)}, \cdots, \frac{f_k(x) + (xD_kx)^{\frac{1}{2}}}{h_k(x)}\right)$$
$$subject\ to\ g(x) \leq 0, x \in X,$$

where X is an open convex subset of R^n , each $f_i :\to R$, $h_i : X \to R, i = 1, \cdots, K$, $g : X \to R^m$ are differentiable and $D_i, i = 1, \cdots, K$ are symmetric positive semidefinite matrices. Let X_0 denote the set of feasible solutions of problem (P). We asumme that X_0 is compact and $h_i(x) > 0$ on $X_0, i = 1, \cdots, K$. All vectors are considered to be column vectors. For simplicity, we avoid the use of the superscript t over a vector to label it as row vector. For instance, instead of writing

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$x_t D_t x$, etc
A sufficent optimality theorem for a feasible solution of (P) to be properly efficient is given. A dual problem for (P) is considered and certain duality theorems are obtained under the ρ-convexity assumptions.

2. Preliminaries

Now we give the definitions and results needed in later in later sections.

Definition 2.1. *A feasible solution $x^0 \in X_0$ is said to be an efficient solution of (P) if there exists no other feasible solution $x \in X_0$ such that*

$$F_i \leq F_i(x^0), i = 1, \cdots, K, F(x) \neq F(X^0).$$

Definition 2.2. *A feasible solution $x^0 \in X^0$ is to be properly efficient for (P) if it is efficient for (P) and if there exists a scalar $M > 0$ such that for each i,*

$$[F_i(x^0) - F_i(x)]/[F_j(x) - F_j(x^0)] \leq M$$

for some j, $F_j(x) > F_j(x^0)$ whenever x is feasible for (P) and $F_i(x) < F_i(x^0)$.

Lemma 2.1 [6]. *Let D be an n, x, n real,symmetric,positive semidefinite matrix. Then, for any $x \in R^n, y \in R^n$,*

$$xDy \leq (xDx)^{\frac{1}{2}}(yDy)^{\frac{1}{2}}.$$

Lemma 2.2[2]. *Let $x^0 \in X_0$. If (x^0, y^0) is properly efficient for the following multiobjective optimization problem (P^1) with $y = y^0$, where $y_i^0 = [f_i(x^0) + (x^0 D_i x^0)^{\frac{1}{2}}]/h_i(x_0), i = 1, \cdots, K$, then x^0 is properly efficient for (P).*

$$(P^1)\ Minimize\ [f_1(x) + (xD_1x)^{\frac{1}{2}} - y_1 h_1(x), \cdots, f_k(x) + (xD_kx)^{\frac{1}{2}} - y_k h_k(x)]$$
$$subject\ to\ g(x) \leq 0, x \in X, y \in R^k.$$

Theorem 2.1 [2]. *Suppose that x^0 is properly efficient soluyion of (P) and the Z^0 is empty. Then there exists a* $\lambda_i > 0, i = 1, \cdots, k$,

$$\sum_{i=1}^{k} \lambda_i = 1, y_i^0 \geq 0, i = 1, \cdots, k, v \in R^m, v_j^0 \geq 0, w_i^0 \in R^n, i = 1, \cdots, k$$

such that

$$\begin{aligned}
&\sum_{i=1}^{k} \lambda_i[\nabla f_i(x^0) + D_i w_i^0 - \nabla y_i^0 h_i(x^0)] + \nabla v^0 g(x^0) = 0 \\
&v^0 g(x^0) \geq 0, \\
&w_i^0 D_i w_i^0 \leq 1, i = 1, \cdots, k \\
&(x^0 D_i x^0)^{\frac{1}{2}} = x^0 D_i w_i^0, i = 1, \cdots, k.
\end{aligned}$$

For a feasible solution x^0 of (P), following Mond and Schechter [9], we define $Z^0 = \cup_{i=1}^{K} Z_i^0$, where

$$\begin{aligned}
Z_i^0 = \{z : \; & z\nabla g_j(x^0) \leq 0 \; for \; all \; j \in \mathrm{Q} \; and \\
& z[\nabla f_i(x^0) - \nabla y_i^0 h_i(x^0)] + zD_i x^0 / x^0 D_i x^0 \; < 0 \; if \; x^0 D_i x^0 \; > 0 \;, \\
& z[\nabla f_i(x^0) - \nabla y_i^0 h_i(x^0)] + (zD_i z) \; < 0 \; if \; x^0 d_i x^0 = 0\}
\end{aligned}$$

for $i = 1, \cdots, k$, where Q$= \{j : g_j(x^0) = 0\}$ and

$$y_i^0 = \frac{f_i(x^0) + (x^0 D_i x^0)^{\frac{1}{2}}}{h_i(x^0)}, i = 1, \cdots, k.$$

Definition 2.3. *Let f be a real valued difficientable function defined on a subset X of R^n. Then f is said to be ρ-convex if there exists some real number ρ such that for each $x, u \in X$,*

$$f(x) - f(y) \geq (x - u)\nabla f(u) + \rho\|x - u\|^2.$$

3. Sufficient Optimality Theorem

Now we establish a sufficient optimality theorem for (P) under the ρ-convex assumptions

Theorem 3.1. *Suppose that there exists a feasible solution x^0 of (p) and a sclar $\lambda_i > 0, i = 1, \cdots, k, \sum_{-=1}^{k} \lambda_i = 1, i = 1, \cdots, k, v^0 \in R^m, v^0 \geq 0, w_i^0 \in R^n, i = 1, \cdots, k$ with*

$$y_i^0 = \frac{f_i(x^0) + (x^0 D_i x^0)^{\frac{1}{2}}}{h_i(x^0)}, i = 1, \cdots, k \textit{ such that}$$

$$(1) \qquad \sum_{i=1}^{k} \lambda_i [\nabla f_i(x^0) + d_i w_i^0 - \nabla y_i^0 h_i(x^0)] + \nabla v^0 g(x^0) = 0$$

$$(2) \qquad v^0 g(x^0) = 0,$$

$$(3) \qquad w_i^0 D_i w_i^0 \leq 1, i = 1, \cdots, k,$$

$$(4) \qquad (x^0 D_i x^0)^{\frac{1}{2}} = x^0 D_i w_i^0, i = 1, \cdots, k.$$

Further suppose that f_i is ρ_i-convex, $-h_i$ is ρ_i^*-convex,$i = 1, \cdots, K$ and g_i is ρ_j^{**}-convex,$j = 1, \cdots, m$ and that

$$\sum_{i=1}^{k} (\lambda_i \rho_i - \lambda_i y_i^0 \rho_i^*) + \sum_{j=1}^{m} v_j^0 \rho_j^{**} \geq 0.$$

Then x^0 is a properly efficient solution of (P).

Proof. By (1),(5) and theρ-convexity assumptions, we have

$$0 \geq \sum_{i=1}^{K} \lambda_i [f_i(x) - f_i(x^0)] + \sum_{i=1}^{K} \lambda (x - x^0) D_i w_i^0$$
$$- \sum_{i=1}^{K} \lambda_i y_i^0 [h_i(x) - h_i(x^0)] + \sum_{i=1}^{m} v_j^0 [g_j(x) - g_j(x^0)].$$

By (2), $v_j^0 g_j(x^0) = 0$ and since x is feasible, $v_j g_j(x) \leq 0,\ j = 1, \cdots, m$. Hence (6) reduces to

$$0 \geq \sum_{i=1}^{K} \lambda_i [f_i(x) - f_i(x^0)] + \sum_{i=1}^{K} \lambda (x - x^0) D_i w_i^0 - \sum_{i=1}^{K} \lambda_i y_i^0 [h_i(x) - h_i(x^0)].$$

By (3), (4) and Lemma2.1, we have

$$0 \le \sum_{i=1}^{K} \lambda_i[f_i(x) - f_i(x^0)] + \sum_{i=1}^{k} \lambda_i[(xD_i x)^{\frac{1}{2}} - (x^0 D_i x^0)^{\frac{1}{2}}] \\ - \sum_{i=1}^{k} \lambda_i y_i^0 [h_i(x) - h_i(x^0)].$$

Hence we have

$$\sum_{i=1}^{k} \lambda_i [f_i(x^0) + (x^0 D_i x^0)^{\frac{1}{2}} - y_i^0 h_i(x^0)] \\ \le \sum_{i=1}^{k} \lambda_i [f_i(x) + (xD_i x)^{\frac{1}{2}} - y_i^0 h_i(x)].$$

By Theorem 1 in [7], (x^0, y^0) is properly efficuent for (p^1).
By Lemma2.2, x^0 is a properly efficient solution of (P).

4. Duality Theorems

Now we give the dual problem (D) for (P). Maximize $G(s, v, y, w_1, \cdots, w_k) = y = (y_1, \cdots, y_k)$ subject to

$$\sum_{i=1}^{k} \lambda_i [\nabla f_i(s) + D_i w_i - \nabla y_i h_i(s)] + \nabla v^t g(s) = 0, \tag{7}$$

$$f_i(s) + (sD_i s)^{\frac{1}{2}} - y_i h_i(s) \ge 0, i = 1, \cdots, k, \tag{8}$$

$$w_i D_i w_i \le 1, i = 1, \cdots, k, \tag{9}$$

$$(sD_i s)^{\frac{1}{2}} = sD_i w_i, i = 1, \cdots, k \tag{10}$$

$$vg(s) \ge 0, \tag{11}$$

$$v \ge 0, y \ge 0, \tag{12}$$

where $\lambda_i > 0, i = 1, \cdots, k$ and $\sum_{i=1}^{k} \lambda_i = 1$.

Theorem 4.1. *Let be any feasible solution of (P) and $(s, v, y, w_1, \cdots, w_k)$ be any feasible sulution of (D) for any $\lambda > 0$. Suppose that f_i is ρ_i-convex, $-h_i$ is ρ_i^*-convex, $i = 1, \cdots, k$, and g_j is ρ_j^{**}-convex, $j = 1, \cdots, m$ and that*

$$\sum_{i=1}^{k}(\lambda_i\rho_i - \lambda_i y_i \rho_i^*) + \sum_{j=1}^{m} v_j \rho_j^{**} \geq 0.$$

Then the following does not holds;

$$\frac{f_i(x) + (xD_ix)^{\frac{1}{2}}}{h_i(x)} \leq y_i \quad \text{for all} \quad i = 1, \cdots, k$$

and

$$\frac{f_j(x) + (xD_jx)^{\frac{1}{2}}}{h_j(x)} < y_j \quad \text{for some} \quad j.$$

Proof.. Suppose that the following holds;

$$\frac{f_i(x) + (xD_ix)^{\frac{1}{2}}}{h_i(x)} \leq y_i \text{ for all } i = 1, \cdots, k$$

and

$$\frac{f_j(x) + (xD_jx)^{\frac{1}{2}}}{h_j(x)} < y_j \text{ for some } j.$$

Then by (8),(11) and (12), we have

$$
\begin{aligned}
0 &> \sum_{i=1}^{k} \lambda_i [f_i(x) + (xD_ix)]^{\frac{1}{2}} - y_i h_i(x)] \\
&\quad - \sum_{i=1}^{k} \lambda_i [f_i(s) + (sD_is)^{\frac{1}{2}} - y_i h_i(s)] + vg(x) - vg(s) \\
&\geq \sum_{i=1}^{k} \lambda_i [(x-s)\nabla f_i(s) + \rho_i \|x-s\|^2] \\
&\quad - \sum_{i=1}^{k} \lambda_i [(x-s)\nabla y_i h_i(s) + \rho_i^* \|x-s\|^2] \\
&\quad + \sum_{j=1}^{m} [(x-s)\nabla v_i g_i(s) + v_j^{**} \rho_j \|x-s\|^2] \\
&\quad + \sum_{i=1} \lambda_i [(xD_ix)^{\frac{1}{2}} - (sD_is)^{\frac{1}{2}}] \\
&\quad (by\ the\ \rho - convexity\ assumptions) \\
&\geq - \sum_{i=1}^{k} \lambda_i (x-s) D_i w_i + \sum_{i=1}^{k} \lambda_i [(xD_ix)^{\frac{1}{2}} - (sD_is)^{\frac{1}{2}}] \\
&\quad + [\sum_{i=1}^{k} (\lambda_i \rho_i - \lambda_i y_i \rho_i^*) + \sum_{i=1}^{m} v_j \rho_j^{**}] \|x-s\|^2 \\
&\quad (by\ (7),\ (9),\ (10)\ and\ Lemma\ 2.1) \\
&\geq [\sum_{i=1}^{k} (\lambda_i \rho_i - \lambda_i y_i \rho_i^{**}) + \sum_{j=1}^{m} v_j \rho_j^{**}] \|x-s\|^2 \\
&\geq 0.
\end{aligned}
$$

This is a contradiction. Hence the result holds.

Theorem 4.2. *Suppose that x^0 is a properly efficient solution of (P) and the set Z^0 is empty. Then exists a feasible solution $(x^0, v^0,$- $w_1^0, \cdots, w_k^0)$ of (D) for some $\lambda > 0$. Futhermore suppose that f_i is ρ_i-convex, $-h_i$ is ρ_i^*-convex,$i = 1, \cdots, k$, and ρ_j^{**}-convex,$j = 1, \cdots, m$*

and that

$$\sum_{i=1}^{k}(\lambda_i\rho_i - \lambda_i y_i\rho_i^*) + \sum_{j=1}^{m} v_j\rho_j^{**} \geq 0.$$

Then $(x^0, v^0, y^0, w_1^0, \cdots, w_k^0)$ is a properly efficient solution of (D) and their respective extreme values are equal.

Proof. By Theorem 2.1,$(x^0, v^0, y^0, w_1^0, \cdots, w_k^0)$ is feasible for (D). By Theorem 4.1, their respective extreme values are equal. Following Theorem4 in [15], $(x^0, v^0, y^0, w_1^0, \cdots, w_k^0)$ is a properly efficient solution of (D).

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