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NOTES ON PRIMARY IDEALS

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We know that a primary ideal of a commutative ring R is defined to be an ideal of R such that if $xy \in I$ and $x \notin I$,then $y^n \in I$ for some positive integer n. B.S. Chew and J. Neggers extended the concept to general rings in their paper[1].

In this paper we will give slightly different definitions of the strongly primary ideal of B.S. Chew and J. Neggers. We will call this *w-strongly primary ideal.* We will show that every w-strongly primary ideal is primary ideal in a commutative ring and a matrix ring of w-strongly primary ring is also w-strongly primary ring. Through this paper we assume that R is a ring with identity and every R- module M is unitary left R-module.

We recall the definitions of primary and strongly primary ideals of B.S. Chew and J. Neggers.

DEFINITION [1]. *Suppose R is a ring. An ideal I of R is called left primary if there is a faithful indecomposable R/I-module M. Moreover if M is both Artinian and noetherian R/I-module, then I is called left strongly primary.*

It is known that every strongly primary ideal is a primary ideal in usual sense in a commutative ring and every primary ideal is a left primary ideal[1]. Usually we call a ring R left primary and left strongly primary if 0 is left primary and strongly primary ideal respectively. Since the integer ring $\mathbb{Z}$ has no faithful noetherian and artinian $\mathbb{Z}$-module,$\mathbb{Z}$ is not strongly primary. Thus we know that primeness does not imply strongly primariness. Either strongly primariness does not imply primeness because $9\mathbb{Z}$ is a strongly primary ideal of an integer ring $\mathbb{Z}$ but not prime.

But we have the following propositions easily.

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PROPOSITION 1. *Let R be a commutative principal ideal domain. Then every nontrivial prime ideal is a strongly primary ideal.*

Proof. Since R is a commutative principal domain,every nontrivial principal prime ideal I is maximal. So R/I is a field and clearly strongly primary.

PROPOSITION 2. *If R is a semisimple primary ring,then R is strongly primary (in fact R is primitive).*

Proof. Let M be a faitheful indecomposable R-module. Since R is semisimple, M is semisimple. So M is simple because M is indecomposable. Thus R has a faithful indecomposable artinian and noetherian.

PROPOSITION 3. *If a left artinian ring R has no nontrivial idempotents, then R is strongly primary.*

Proof. Let $M = {}_RR$. Then $End_R(M) \cong R$. Since R has no nontrivial idempotents, M is indecomposable and $M = R$ is artinian and noetherian R-module.

PROPOSITION 4. *If R is semisimple, the intersection of strongly primary ideals is zero.*

Proof. Let $R = \bigoplus_{\imath \in I} I_\imath$ where $I_\imath$ is minimal left ideal and $J_\imath = ann_\ell(I_\imath) = \{r \in R \mid rI_\imath = 0\}$. Clearly $J_\imath$ is two sided ideal and strongly primary for $I_\imath$ is a faithful indecomposable artinian and noetherian $R/J_\imath$-module.Clearly $\bigcap_{\imath \in I} J_\imath = \{0\}$

Also we know that if R is a right Goldie ring, then the intersection of all primary ideals is zero by similar method.

The following theorem shows that if R is a left artinian primary ring and R have an injective left nonzero ideal,then R is a left strongly primary ring.

THEOREM 1. *Let R be a left artinian and R have an injective left nonzero ideal. Then if R is a left primary ring, R is a left strongly primary ring.*

Proof. Suppose L is an injective left ideal. Then L is a direct summand of R,that is $R = L \oplus L'$ for a suitable left ideal L' of R. Since R is left artinian,we can refine this decomposition into an indecomposable direct decomposition of R. Let $R = I \oplus I'$ where a left ideal I is a direct summand of L(so I is injective) and I is indecomposable left R-module. Since I is left artinian, I contains a simple left ideal J. Then I is the injective envelope of J for I is indecomposable and injective. Thus J is a unique simple left ideal of I. Since R is primary,R has a faithful indecomposable R-module M. Then there exists an element m in M such that $Jm \neq 0$ for $JM \neq 0$. We can define an R-module homomorphism Φ_m from I into M as follows $\Phi_m(a) = am$. Then $Ker\Phi_m = \{a \mid am = 0\}$ does not contain J. So $Ker\Phi_m = \{0\}$ for J is the unique minimal left ideal of I. Thus Φ_m is a monomorphism and $Im \cong I$ is an injective submodule of M. Moreover Im is a direct summand of M by injectiveness of Im. Clearly $Im \cong M$. Thus R has a faithful indecomposable artinian and noetherian R-module M for I is left artinian and noetherian.

We define w-strongly primary ideal as following.

DEFINITION. *An ideal I of a ring R is called w-strongly primary ideal if there exits a faithful R/I-module M such that $End_{R/I}(M)$ is local ring and its Jacobson radical is nil ideal.*

We know that if M is indecomposable artinian and noetherian R-module, $End_R(M)$ is local ring and its Jacobson radical is nilpotent[2]. Thus every strongly primary ring is w-strongly primary.

The following theorems show that every commutative w-strongly primary ring is primary and a matrix ring of w-strongly primary ring is also w-strongly primary ring.

THEOREM 2. *Let R be a commutative ring. If R is w-strongly primary, R is primary.*

Proof. Let M be a faithful R-module and $S = End_R(M)$ be local and its Jacobson radical be nil. We imbedds R in S via $T_a(m) = am$ (in fact a is mapped into T_a). Let $ab = 0$ and $b \neq 0$ in R. Then $T_aT_b = T_{ab} = 0$ and $T_b \neq 0$. So $T_a \in rad(S)$. Since $rad(S)$ is nil,$(T_a)^n = 0$ for some n. Thus $(T_a)^n = 0$ implies $a^nM = 0$ for $(T_a)^n = T_a^n$. Since M is a faithful R-module,$a^n = 0$.

THEOREM 3. *R is a w-strongly primary ring iff $M_n(R)$ is a w-strongly primary ring where $M_n(R)$ is (n, n) matrix ring over R.*

Proof. If M is a faithful R-module such that $End_R(M)$ is local and its Jacobson radical is nil. Let $N = M \oplus \cdots \oplus M$ (n-copies) as a direct sum of groups. We define $M_n(R)$-action as following ;

$$(r_{ij})(m_1, \dots m_i, \dots m_n) \stackrel{\text{def}}{=} (\dots, \sum_{j=1}^{n} r_{ij} m_j, \dots)$$

Then N is a faithful $M_n(R)$-module. We will prove that $End_R(M) \cong End_{M_n(R)}(N)$. At first we can define a ring homomorpism Ψ from $End_R(M)$ into $End_{M_n(R)}(N)$ as following ;

$$\Psi(\sigma)(m_1, \dots, m_i, \dots, m_n) \stackrel{\text{def}}{=} (\sigma(m_1), \dots, \sigma(m_i), \dots, \sigma(m_n))$$

for every $\sigma \in End_R(M)$. By simple calculation, we know that $\Psi(\sigma)$ is an element of $End_{M_n(R)}(N)$ and Ψ is a ring homomorphism. On the other hand if τ is any $M_n(R)$-module homomorphism of N.

Since

$$\begin{aligned} \tau(0, \dots, m_i, 0, \dots, 0) &= \tau(E_{ii}(0, \dots, m_i, 0, \dots, 0) \\ &= E_{ii}\tau(0, \dots, m_i, 0, \dots, 0) \\ &= E_{ii}(m_1', \dots, m_i', \dots, m_n') \\ &= (0, \dots, 0, m_i', 0, \dots, 0), \end{aligned}$$

we have $\tau(0 \dots, m_i, 0, \dots, 0) = (0, \dots, m_i', 0, \dots, 0)$ where E_{ij} is the matrix whose element of i-th row and j-th column is 1 and otherwise is 0. For each i, we can define σ_i as $\sigma_i(m) = \pi_i \tau \iota_i(m)$ where ι_i is i-th injection from M into N and π_i is i-th projection from N into M. Then clearly σ_i is R-module homomorphism of M.

Since

$$\begin{aligned} \sigma_i(m) &= \pi_i \tau \iota_i(m) \\ &= \pi_i \tau(E_{ij} \iota_j(m)) \\ &= \pi_i E_{ij} \tau \iota_j(m) \\ &= \pi_i E_{ij}(0, \dots, \sigma_j(m), \dots, 0, \dots, 0) \\ &= \sigma_j(m), \end{aligned}$$

we have $\sigma_i = \sigma_j = \sigma$ for every $i \neq j$. Thus $\tau = \Psi(\sigma)$. It is clear that Ψ is one to one. Hence Ψ is an isomorphism and $End_R(M) \cong End_{M_n(R)}(N)$.

Conversely N is a faithful $M_n(R)$-module. Define $N_i = E_{ii}N$. Then $N = N_1 \oplus, \ldots, \oplus N_n$ as a direct sum of abelian groups and $N_i \cong N_j$ for $i \neq j$. Each N_i is an R-module via $rn = rE_{ii}n$ for $n \in N_i$. Clearly $M = N_i$ is a faithful R-module and $End_R(M) \cong End_{M_n(R)}(N)$. Thus theorem is proved.

References

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