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SIMPLER AXIOMATIC SYSTEMS OF LATTICE TOPOLOGIES

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In 1984, P. Z. Wang [3] gave several axiomatic systems of lattice topology, and obtained some results.

In this paper, we simplify some axiomatic systems. In particular, the 7 axioms of net convergence relation are simplified into 5 axioms of it.

DEFINITION 1. Let $(L;\geq)$ be a complete lattice with the greatest element 1 and the least element 0. Then $(L;\geq)$ is said to be dual if there is a map $c : L \to L$ such that

(1) $(\alpha^c)^c = \alpha$ for all $\alpha \in L$,

(2) $(\alpha \vee \beta)^c = \alpha^c \wedge \beta^c$ and $(\alpha \wedge \beta)^c = \alpha^c \vee \beta^c$ for all $\alpha, \beta \in L$.

Throughout this paper, L always means a complete dual lattice and we note that the relation "$\leq$" is the inverse of "$\geq$".

For any nonempty subsets A and B of L, define $A \sim B$ if and only if $\forall \alpha \in A\ \exists \beta \in B$ such that $\alpha \geq \beta$, and $\forall \beta \in B\ \exists \alpha \in A$ such that $\beta \geq \alpha$.

DEFINITION 2. A nonempty subset R of L is called a filter if

(3) $\forall \alpha, \beta \in L\ \alpha \geq \beta$ and $\beta \in R \Rightarrow \alpha \in R$,

(4) $\forall \alpha, \beta \in R\ \exists \gamma \in R$ such that $\alpha \geq \gamma$ and $\beta \geq \gamma$.

Denote by $\tau(L)$ the set of all filters of L. We first give a characterization of filters.

LEMMA 1. *A nonempty subset R of L is a filter of L if and only if it satisfies (3) and*

(5) $\forall \alpha, \beta \in R$, $\alpha \wedge \beta \in R$,

where $\alpha \wedge \beta = \inf\{\alpha, \beta\}$.

Proof. It is sufficient to show that if R satisfies (3) then the conditions (4) and (5) are equivalent. The fact that (5) implies (4) is

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obvious, because $\alpha, \beta \geq \alpha \wedge \beta$. If R satisfies (4), that is, $\forall \alpha, \beta \in R$ $\exists \gamma \in R$ such that $\alpha \geq \gamma$ and $\beta \geq \gamma$, then $\alpha \wedge \beta \geq \gamma$. It follows from (3) that $\alpha \wedge \beta \in R$. This completes the proof.

LEMMA 2. *If $R_t, t \in T$, is an indexed family of filters of L, then $\cap\{R_t : t \in T\}$ is a filter of L.*

Proof. Obvious.

DEFINITION 3. Denote

$$n_L = \{n | n : L \to \tau(L) \text{ such that (n1) – (n3)}\},$$

where

(n1) $n(0) = L$,
(n2) $\beta \in n(\alpha) \Rightarrow (\exists \gamma)((\beta \geq \gamma \geq \alpha)((\forall \delta)(\gamma \geq \delta \Rightarrow \gamma \in n(\delta))),$
(n3) $n(\vee\{\alpha_t : t \in T\}) = \cap\{n(\alpha_t) : t \in T\}$.

Then $n \in n_L$ is called a neighborhood structure of L, $n(\alpha)$ is called a neighborhood system of α, and $N = \cup\{n(\alpha) : \alpha \in L\}$ is called the neighborhood system of L.

THEOREM 1. *If $n \in n_L$ then*
(n4) $\beta \in n(\alpha) \Rightarrow \beta \geq \alpha$.

Proof. By (n2), we know that

$$\beta \in n(\alpha) \Rightarrow (\exists \gamma)(\beta \geq \gamma \geq \alpha) \Rightarrow \beta \geq \alpha.$$

REMARK 1. In [4], the axiomatic system of the neighborhood structure of L was given by (n1) – (n4). But Theorem 1 above shows that it can be defined by (n1) – (n3) only.

DEFINITION 4. Denote

$$r_L = \{\upharpoonright \in \mathcal{P}(\tau(L) \times L) | \upharpoonright \text{ satisfies (r1) – (r5)}\},$$

where

(r1) $(R, 0) \in \upharpoonright \Rightarrow R = L$,
(r2) $\forall \alpha \in L, (\dot{\alpha}, \alpha) \in \upharpoonright$, where $\dot{\alpha} = \{\beta \in L | \beta \geq \alpha\}$,
(r3) $\forall t \in T, (R_t, \alpha) \in \upharpoonright$ and $R \supset \cap\{R_t : t \in T\} \Rightarrow (R, \alpha) \in \upharpoonright$.

Define $\beta[\alpha$ if and only if $(\forall R \in \tau(L))((R, \alpha) \in \upharpoonright \Rightarrow \beta \in R)$.

(r4) $\beta[\alpha \Rightarrow (\exists \gamma)((\beta \geq \gamma \geq \alpha)((\forall \delta)(\gamma \geq \delta \Rightarrow \gamma[\delta))),$
(r5) $\forall t \in T, \beta[\alpha_t \Rightarrow \beta[\bigvee_{t \in T} \alpha_t$.

Then $\upharpoonright \in r_L$ is called a filter convergence relation of L. If $(R, \alpha) \in \upharpoonright$, we say that R converges to α, and is denoted by $R \upharpoonright \alpha$.

PROPOSITION 1. *If* $\upharpoonright \in r_L$ *then* $\forall t \in T, \beta[\alpha_t \Leftrightarrow \beta[\bigvee_{t\in T} \alpha_t$.

Proof. Sufficiency. If $\beta[\bigvee_{t\in T} \alpha_t$ then by (r4) we have

$$(\exists\gamma)((\beta \geq \gamma \geq \bigvee_{t\in T}\alpha_t)((\forall\delta)(\gamma \geq \delta \Rightarrow \gamma[\delta))).$$

Since $\alpha_t \leq \bigvee_{t\in T}\alpha_t \leq \gamma$ for every $t \in T$, it follows that $\gamma[\alpha_t$ for all $t \in T$. Let $R \in \tau(L)$ satisfy $R \upharpoonright \alpha_t$ $(\forall t \in T)$. Then $\gamma \in R$. Combining $\beta \geq \gamma$, we get $\beta \in R$ by (3). This shows that $\beta[\alpha_t$ $(\forall t \in T)$.

Necessity is clear from (r5). The proof is complete.

THEOREM 2. *Let* $\upharpoonright \in r_L$. *Then*

(r6) $R \upharpoonright \alpha$ *and* $R \subset R' \Rightarrow R' \upharpoonright \alpha$.

(r7) $R_t \upharpoonright \alpha$ $(\forall t \in T) \Rightarrow \cap\{R_t : t \in T\} \upharpoonright \alpha$.

Conversely if $\upharpoonright \in \mathcal{P}(\tau(L) \times L)$ *satisfies (r1), (r2), (r4), (r5), (r6) and (r7) then (r3) holds, and hence* $\upharpoonright \in r_L$.

Proof. (r6) follows directly from (r3). By Lemma 2 we know that $\cap\{R_t : t \in T\}$ is a filter of L. Hence (r7) holds by (r3).

Conversely let $\upharpoonright \in \mathcal{P}(\tau(L) \times L)$ satisfy (r1), (r2), (r4), (r5), (r6) and (r7). Suppose $R_t \upharpoonright \alpha$ $(\forall t \in T)$ and let $R \supset \cap\{R_t : t \in T\}$. Then by (r7), $\cap\{R_t : t \in T\} \upharpoonright \alpha$. It follows from (r6) that $R \upharpoonright \alpha$. Thus (r3) holds.

REMARK 2. The axiomatic system of filter convergence relation in [3] consisted of (r1), (r2), (r4), (r5), (r6) and (r7). Theorem 2 above shows that the above conditions are equivalent to the conditions (r1), (r2), (r3), (r4) and (r5). Hence Definition 4 is a simplification of axiomatic system of filter convergence relation in [3].

In the proof of the following proposition, P. Z. Wang have been used the notion of neighborhhood system. But we prove it without the notion of neighborhood system.

PROPOSITION 2 ([3, PROPOSITION 2.3]). *If* $\upharpoonright \in r_L$, *then*

(r8) $R \upharpoonright \alpha$ *and* $\alpha' \geq \alpha \Rightarrow R \upharpoonright \alpha'$.

Proof. Denote $\tau(\alpha) = \{R \in \tau(L) : R \upharpoonright \alpha\}$. Observe that $\beta[\alpha$ is equivalent to $\beta \in \cap\{R : R \in \tau(\alpha)\}$. By Proposition 1, we have

$$\bigcap_{t\in T} \cap \{R : R \in \tau(\alpha_t)\} = \cap\{R : R \in \tau(\bigvee_{t\in T}\alpha_t)\}.$$

If $\alpha' \geq \alpha$ then $\alpha' \vee \alpha = \alpha'$. Thus

$$\begin{aligned}&(\cap\{R : R \in \tau(\alpha)\}) \cap (\cap\{R : R \in \tau(\alpha')\})\\&= \cap\{R : R \in \tau(\alpha' \vee \alpha)\}\\&= \cap\{R : R \in \tau(\alpha')\}.\end{aligned}$$

Hence $\cap\{R : R \in \tau(\alpha)\} \supset \cap\{R : R \in \tau(\alpha')\}$. If $R \upharpoonright \alpha$ then

$$R \supset \cap\{R : R \in \tau(\alpha)\} \supset \cap\{R : R \in \tau(\alpha')\}.$$

It follows from (r3) that $R \upharpoonright \alpha'$. This completes the proof.

A binary relation $\geq$ directs a set T if T is non-void and

(a) if $m, n, p \in T$ are such that $m \geq n$ and $n \geq p$, then $m \geq p$;
(b) if $m \in T$, then $m \geq m$;
(c) if $m, n \in T$, then there is $p \in T$ such that $p \geq m$ and $p \geq n$.

A directed set is a pair $(T, \geq)$ such that $\geq$ directs T. A net is a pair $(S, \geq)$ such that S is a function and $\geq$ directs the domain of S.

DEFINITION 5. Let $(T, \geq)$ be a directed set, $(D_t, \geq_t)$ a directed set for each t in T, and let $\Pi = \prod_{t \in T} D_t \times T$. Define a map $\omega_\Pi : \Pi \to L$ by $\omega_\Pi(f, t) = \omega_t(f(t))$ for each $(f, t) \in \Pi$, where $\omega_t : D_t \to L$ $(t \in T)$ is a function. Then we say that ω_Π is the product net generated by $\{(D_t, \geq_t) : t \in T\}$.

Let $W = W(L)$ be a set of nets in L. Consider the following conditions:

(ω1) If D is a filter of L, then $i_D \in W$, where i_D is the identity map of D.
(ω2) Let D be a directed set and let D' be a directed subset of D. If $\omega : D \to L$ is a net, then $\omega' = \omega|D'$ is also a net, where $\omega|D'$ is the restriction of ω on D'.
(ω3) Let $\omega_t : D_t \to L$ ($\forall t \in T$, a directed set). If $\omega_t \in W$ $(t \in T)$, then the product net $\omega_\Pi \in W$.

DEFINITION 6. If $W = W(L)$ satisfies the conditions (ω1) - (ω3), then we say that W is sufficient.

Given a net $\omega : D \to L$ ($(D, \geq)$ is a directed set), denote

$$F(\omega) = \{\alpha \in L : \omega(d) \leq \alpha \text{ eventually}\},$$

where "$\omega(d) \leq \alpha$ eventually" means that there exists $d_0 \in D$ such that $d \leq d_0$ implies $\omega(d) \leq \alpha$.

DEFINITION 7. Let W be a set of nets in L, which is sufficient. Denote

$$l_L = \{\searrow | \searrow \subset W \times L \text{ such that } (l1) \text{ - } (l5)\},$$

where

($l1$) $(\omega, 0) \in \searrow \Rightarrow \omega(d) = 0$ eventually,

($l2$) $\omega(d) \leq \alpha$ eventually $\Rightarrow (\omega, \alpha) \in \searrow$,

($l3$) $(\omega_t, \alpha) \in \searrow$ $(\forall t \in T)$ and $F(\omega) \supset \bigcap_{t \in T} F(\omega_t) \Rightarrow (\omega, \alpha) \in \searrow$.

If for every $\omega \in W$

$$(\omega, \alpha) \in \searrow \Rightarrow \omega(d) \leq \beta \text{ eventually},$$

then we say that β covers α.

($l4$) If β covers α, then

$$(\exists \gamma)((\beta \geq \gamma \geq \alpha)((\forall \delta)(\gamma \geq \delta \Rightarrow \gamma \text{ covers } \delta))),$$

($l5$) If β covers α_t $(t \in T)$, then β covers $\bigvee_{t \in T} \alpha_t$.

Then $\searrow \in l_L$ is called a (W) net convergence relation on L. If $(\omega, \alpha) \in \searrow$, we say that ω converges to α, denoted by $\omega \searrow \alpha$.

PROPOSITION 3. *If $\searrow \in l_L$, then β covers α_t $(t \in T)$ if and only if β covers $\bigvee_{t \in T} \alpha_t$.*

Proof. Necessity follows from ($l5$).

Sufficiency. Suppose β covers $\bigvee_{t \in T} \alpha_t$. Then by ($l4$),

$$(\exists \gamma)((\beta \geq \gamma \geq \bigvee_{t \in T} \alpha_t)((\forall \delta)(\gamma \geq \delta \Rightarrow \gamma \text{ covers } \delta))).$$

Since $\gamma \geq \bigvee_{t \in T} \alpha_t \geq \alpha_t$ $(t \in T)$, taking $\delta = \alpha_t$ we have that γ covers α_t $(t \in T)$, and hence β covers α_t $(t \in T)$. This completes the proof.

THEOREM 3. *Let $\searrow \in l_L$. Then $\searrow$ satisfies the following conditions:*

($l6$) *If $\omega_t \searrow \alpha_t \in L$ $(t \in T)$ and $\omega^* \searrow \alpha$, where T is a directed set and $\omega^* : T \to L$ is defined by $\omega^*(t) = \alpha_t$ for each $t \in T$, then $\omega_\Pi \searrow \alpha$;*

($l7$) *If $\omega \searrow \alpha$ and $\alpha' \geq \alpha$, then $\omega \searrow \alpha'$;*

($l8$) *Let ω be a net. If for each subnet ω' of ω, there is a subnet ω'' of ω' such that $\omega'' \searrow \alpha$, then $\omega \searrow \alpha$.*

Proof. Denote $W(\alpha) = \{\omega \in W : \omega \searrow \alpha\}$ and

$$\Gamma(\alpha) = \{\beta \in L : \beta \text{ covers } \alpha, \text{ and } (\forall\delta)(\beta \geq \delta \Rightarrow \beta \text{ covers } \delta)\}.$$

The fact that β covers α is equivalent to

$$\beta \in \cap\{F(\omega) : \omega \in W(\alpha)\}.$$

Obviously ($l4$) means that

$$\overline{\cap\{F(\omega) : \omega \in W(\alpha)\}} \sim \Gamma(\alpha).$$

Therefore $\omega \searrow \alpha \Leftrightarrow F(\omega) \supset \Gamma(\alpha)$. We now show that $\omega_\Pi \searrow \alpha$. Since $\omega^* \searrow \alpha$, therefore $F(\omega^*) \supset \Gamma(\alpha)$. Let $\gamma \in \Gamma(\alpha)$. Then there is $t_0 \in T$ such that $\omega^*(t) = \alpha_t \leq \gamma$ whenever $t \leq t_0$. So γ covers $\omega^*(t) = \alpha_t$ ($t \leq t_0$). It follows from $\omega_t \searrow \alpha_t$ that there is $d_t^* \in D_t$ such that $\omega_t(d_t) \leq \gamma$ ($t \leq t_0$) whenever $d_t \leq d_t^*$, where D_t is the domain of a net ω_t. If we take (f_0, t_0) in $\Pi = (\prod_{t\in T} D_t) \times T$, where

$$f_0(t) = \begin{cases} d_t^*, & \text{if } t \leq t_0, \\ \text{any element in } D_t, & \text{otherwise,} \end{cases}$$

then for every $(f,t) \in \Pi$ with $(f,t) \leq (f_0, t_0)$, we have $t \leq t_0$ and $f(t) \leq f_0(t) = d_t^*$, where $f(t) \in D_t$. Thus $\omega_\Pi(f,t) = \omega_t(f(t)) \leq \gamma$. This proves that $F(\omega_\Pi) \supset \Gamma(\alpha)$. Therefore $\omega_\Pi \searrow \alpha$, and ($l6$) is true.

We notice that Proposition 3 is equivalent to

(6) $\bigcap_{t\in T} \cap \{F(\omega) : \omega \in W(\alpha_t)\} = \cap\{F(\omega) : \omega \in W(\bigvee_{t\in T} \alpha_t)\}$.

If $\alpha' \geq \alpha$ then $\alpha' \vee \alpha = \alpha'$. By (6) we have

$$\begin{aligned}&(\cap\{F(\omega) : \omega \in W(\alpha)\}) \cap (\cap\{F(\omega) : \omega \in W(\alpha')\}) \\ &= \cap\{F(\omega) : \omega \in W(\alpha' \vee \alpha)\} \\ &= \cap\{F(\omega) : \omega \in W(\alpha')\}.\end{aligned}$$

Hence $\cap\{F(\omega) : \omega \in W(\alpha)\} \supset \cap\{F(\omega) : \omega \in W(\alpha')\}$. If $\omega \searrow \alpha$ then $\omega \in W(\alpha)$. Therefore

$$F(\omega) \supset \cap\{F(\omega) : \omega \in W(\alpha)\} \supset \cap\{F(\omega) : \omega \in W(\alpha')\}.$$

It follows from ($l3$) that $\omega \searrow \alpha'$, which proves ($l7$).

In order to prove ($l8$), let $\omega : D \to L$ be a net such that for each subnet ω' of ω, there is a subnet ω'' of ω' satisfying $\omega'' \searrow \alpha$, but ω does not converge to α. Then by ($l3$), we have $F(\omega) \not\supset \cap\{F(\omega) : \omega \in W(\alpha)\}$. Hence there exists $\beta_0 \in \cap\{F(\omega) : \omega \in W(\alpha)\}$, but $\beta_0 \notin F(\omega)$. The latter means that

(7) $(\forall d \in D)((\exists d' \in D)((d' \leq d)(\omega(d') \not\leq \beta_0)))$.

Let $D' = \{d' \in D : \omega(d') \not\leq \beta_0\}$. Then $(D', \geq)$ is a directed subset of $(D, \geq)$. Since W is sufficient, it follows that $\omega' \in W$. By (7) we have

(8) $\omega'(d) \not\leq \beta_0$ for all $d \in D'$.

For each subnet ω'' of ω'. we will prove that ω'' does not converge to α. If $\omega'' \searrow \alpha$ then $\omega'' \in W(\alpha)$. Therefore

$$F(\omega'') \supset \cap\{F(\omega) : \omega \in W(\alpha)\}.$$

Hence

(9) $\beta_0 \in F(\omega'')$.

Let $\bar{\omega} : \bar{D} \to L$ be a subnet of ω' such that there is a map $N : \bar{D} \to D'$ satisfying the axioms:

(a) $\bar{\omega} = \omega' \circ N$, i.e., $(\forall d \in \bar{D})\ (\bar{\omega}(d) = \omega'(N(d)))$,

(b) $\forall d' \in D'\ \exists \bar{d} \in \bar{D}$ such that $(\forall d \in \bar{D})(d \leq \bar{d} \Rightarrow N(d) \leq d')$.

Thus (9) implies that

$$(\exists d_0 \in \bar{D})((\forall d \in \bar{D})(d \leq d_0 \Rightarrow (\bar{\omega}(d) = \omega'(N(d)) \leq \beta_0))).$$

This contradicts to (8), and hence ($l8$) holds. This completes the proof.

THEOREM 4. *If* $\searrow \subset W \times L$ *satisfies* ($l1$) - ($l3$) *and* ($l5$) - ($l7$), *then it satisfies* ($l4$), *and hence* $\searrow \in l_L$.

Proof. It is similar to the final paragraph in the proof of [4; Theorem 4.1.3].

REMARK 3. In [3], the axiomatic system of net convergence relation was given by ($l1$) - ($l3$) and ($l5$) - ($l8$). But we know that, from Theorems 3 and 4, the conditions ($l1$) - ($l3$) and ($l5$) - ($l8$) are equivalent to that of Definition 7. Hence ($l1$) - ($l5$) are a simplication of axiomatic system of net convergence relation in [3].

REMARK 4. Following Definition 7, the non-uniqueness of the net convergence relation, i.e., ($l7$), is a consequence of net convergence axioms.

REMARK 5. Theorems 3 and 4 also show that ($l8$) is a consequence of ($l1$) - ($l3$) and ($l5$) - ($l7$). Hence the axiomatic system of net convergence relation in [3] is not independent.

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