Pusan Kyŏngnam Math. J. 9(1993), No. 2, pp. 271–274

GENERALIZATIONS OF GROTHENDIECK'S THEOREM

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Let $C_p(X)$ denote the space of all continuous real-valued functions on X with the topology of pointwise convergence. M. O. Asanov and N. V. Velichko showed that in [2] if X is countably compact, then the closure of every bounded subset in $C_p(X)$ is compact, and if X is a k-space, then the closure in $C_p(X)$ of every bounded subset is compact. In [1], A. V. Arhangel'skii generalized above theorem as follows, using the concept of functionally generated families:

(1) If a space X is functionally generated by the family $\mathcal{M}_c$ of all its closed subspaces that have a dense σ-countably paracompact subspace, then the closure in $C_p(X)$ of every pseudocompact subspace is compact.

(2) If X is functionally generated by the family $\mathcal{M}_n$ of its own closed subspaces that contain a dense σ-pseudocompact subspace, then the closure in $C_p(X)$ of every contably paracompact subspace is compact.

Moreover, V. V. Tkachuk showed that there is a pseudocompact space X such that $C_p(X)$ has a closed pseudocompact (*a fortiori*, bounded) subspace which is not compact (see [1]).

In this paper, we generalize the second theorem of A. V. Arhangelskii and show that for a separable space X, the closure in $C_p(X)$ of every bounded subset is Eberlein compact. Throughout this paper, we assume that all spaces are Tychonoff spaces.

DEFENITION 1 ([3]). *Let $\mathcal{M}$ be a family of subspaces of X. We say that X is functionally generated by $\mathcal{M}$ if for any discontinuous function $f : X \longrightarrow R$ there is an $A \in \mathcal{M}$ such that $f|_A$ cannot be extended to a continuous real-valued function on X.*

The space $C_p(X)$ is regarded as a subspace of R^X. For a subspace Y of X, we define a map π_Y from $R^X(C_p(X))$ into $R^Y(C_p(Y))$ by $\pi_Y(f) = f|_Y$ for all $f \in R^X$(resp. $C_p(X)$), where $f|_Y$ is the restriction of f on Y. Hereafter, we use the notation $C_p(Y \mid X)$ for the subspace $\pi_Y(C_p(X))$ of $C_p(Y)$.

Received September 25, 1993 .

A subspace Y of X is said to be *P-situated in X* if Y has the property P. We say that a property P is *continuously invariant* [1] if for each P-situated set Y in X and for each continuous map f defined on X, $f(Y)$ is P-situated in $f(X)$.

PROPOSITION 2 ([1]). *Let X be functionally generated by a family $\mathcal{M}$ of its subspaces, and P be a continuous invariant property such that:*

(*) If $Y \in \mathcal{M}$, the closure of any P-situated set in $C_p(Y \mid X)$ is compact.

Then any P-situated set, A, in $C_p(X)$, the closure of A in $C_p(X)$ coincides with the closure in R^X and is compact.

A space X is said to be *$d\sigma$-bounded* ([1]) if X has a dense subspace which is the union of a countable family of bounded sets in X.

THEOREM 3 ([1]). *If X is $d\sigma$-bounded, then every countably paracompact subspace F of $C_p(X)$ is Eberlein compact.*

We now obtain a main result.

THEOREM 4. *If X is functionally generated by a family $\mathcal{M}$ of $d\sigma$-bounded subspaces of X, then the closure of every countably paracompact set is compact.*

Proof. We claim that Proposition 2(*) is satisfied if we take countably paracompactness as the property P. Suppose that $Y \in \mathcal{M}$, $Y \subset X$, and A is an arbitrary countably paracompact subspace of $C_p(Y \mid X)$. Then A is an Eberlein compact subspace in $C_p(Y)$. So $\bar{A} = A$ is compact in $C_p(Y \mid X)$. Thus it follows from Proposition 2 that the closure in $C_p(X)$ of every countably paracompact subspace is compact.

COROLLARY 5 ([1]). *If X is functionally generated by the family $\mathcal{M}_n$ of its own closed subspaces that contain a dense σ-pseudocompact subspace, then the closure in $C_p(X)$ of every countably paracompact subspace is compact.*

Proof. Since every space which contains a dense σ-pseudocompact subspace is $d\sigma$-bounded, the result is immediate from Theorem 4.

LEMMA 6 ([1]). *Let X and Z be spaces, f a map from X into Z, and Y a dense subset of X. Suppose that the restriction of f to each subspace of the form $Y \cup \{x\}$ is continuous, where x is an arbitrary point of X. Then f is continuous.*

A space X is said to be ***k*-separable** ([1]) if it contains a dense σ-compact subspace. A space X is said to be ***functionally complete*** ([1]) if there is a compact set $F \subset C_p(X)$ that separates points in X.

THEOREM 7 [3]. *If X is separable, then $C_p(X)$ is submetrizable.*

LEMMA 8. *If X is separable, then $C_p(X)$ is functionally complete.*

Proof. Since X is separable, $C_p(X)$ is submetrizable. So we see that $C_p(C_p(X))$ is k-separable. Thus $C_p(C_p(C_p(X)))$ is functionally complete, and so $C_p(X)$ is functionally complete.

THEOREM 9. *If X is separable, then the closure of any bounded set in X is Eberlein compact.*

Proof. Since X is separable, X has a dense subset D. Let $\mathcal{M} = \{D \cup \{x\} \mid x \in X\}$. By Lemma 6, X is functionally generated by $\mathcal{M}$. Now, we claim that Propositon 2(*) is satisfied if we take boundedness as the property P. Suppose that $Y = D \cup \{x\} \in \mathcal{M}$ and A is an arbitrary bounded subspace of $C_p(Y \mid X)$. Since for every $Y \in \mathcal{M}$, $C_p(Y \mid X)$ has a countable base, the closure of A is $C_p(Y \mid X)$ is compact. So the closure of any bounded set in $C_p(X)$ is compact. Since every subspace of functionally complete space is functionally complete, the closure of bounded set in $C_p(X)$ is Eberlein compact.

Remark. In [2], M. O. Asanov and N. V. Velichko showed that if X is a k-space, then the closure in $C_p(X)$ of every bounded set is compact.

It is well-known that R^R is separable, but not a k-space (see [4]). From this example, we see that Theorem 9 and the result obtained by M. O. Asanov and N. V. Velichko are independent.

We say that a family $\mathcal{M}$ of subspaces of X is ***strongly functionally generated by*** $\mathcal{M}$ if for each discontinuous real-valued function f on X, there is a $Y \in \mathcal{M}$ such that $f\mid_Y : Y \longrightarrow R$ is discontinuous ([1]).

THEOREM 10. *If X is strongly functionally generated by separable subspaces, then every closure of bounded set in $C_p(X)$ is compact.*

Proof. For any $Y \in \mathcal{M}$, let B be a bounded set in $C_p(Y)$. By Theorem 9, the closure of B in $C_p(Y)$ is Eberlein compact. So the closure of every bounded set in $C_p(X)$ is compact.

COROLLARY 11. *If X has a submetrizable subspace, then the closure of every bounded set in $C_p(X)$ is compact.*

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