Pusan Kyŏngnam Math. J. 9(1993), No 2, pp 261–269

FUZZY IRREDUCIBLE IDEALS IN Γ–RINGS

YOUNG BAE JUN AND CHONG YUN LEE

In 1965, Zadeh [11] introduced the notion of fuzzy sets in a set S as a function from S into $[0,1]$. Rosenfeld [9] applied this concept to the theory of groupoids and groups. In [6], Kumar discussed the fuzzy irreducible ideals in rings. Motivated by the study of Kumar, we discuss, in this paper, the fuzzy irreducible ideals in Γ-rings.

We first review some fuzzy logic concepts. For any fuzzy sets μ and ν in a set S, we define

$$\mu \subseteq \nu \Longleftrightarrow \mu(x) \leq \nu(x) \quad \text{for all } x \in X,$$
$$(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\} \quad \text{for all } x \in X.$$

Let μ be any fuzzy set in a set S. The set

$$\mu_t = \{x \in X : \mu(x) \geq t\}, \quad \text{where } t \in [0,1],$$

is called a level subset of μ.

Let S and S' be any two sets and let $f : S \to S'$ be any function. If μ is any fuzzy set in S, then the fuzzy set ν in S' defined by

$$\nu(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, y \in S', \\ 0 & \text{otherwise,} \end{cases}$$

is called the image of μ under f, denoted by $f(\mu)$. If ν is a fuzzy set in $f(S)$, then the fuzzy set μ in S defined by $\mu(x) = \nu(f(x))$ for all $x \in S$ is called the preimage of ν under f and is denoted by $f^{-1}(\nu)$. A fuzzy set μ in S is said to be f-invariant if

$$f(x) = f(y) \Rightarrow \mu(x) = \mu(y), \quad \text{where } x, y \in S.$$

A fuzzy set μ in a set S has sup property if, for any subset T of S, there exists $x_0 \in T$ such that

$$\mu(x_0) = \sup_{t \in T} \mu(t).$$

Received September 24,1993 .

LEMMA 1 ([9]). *Let f be a function defined on a set S. Then*

(a) $\mu \subseteq f^{-1}(f(\mu))$ *for any fuzzy set* μ *in* S,

(b) $\mu = f^{-1}(f(\mu))$ *provided that* μ *is* f*-invariant for any fuzzy set* μ *in* S,

(c) $\mu_1 \subseteq \mu_2 \Rightarrow f(\mu_1) \subseteq f(\mu_2)$ *for any fuzzy sets* μ_1, μ_2 *in* S,

(d) $f(f^{-1}(\nu)) = \nu$ *for any fuzzy set* ν *in* $f(S)$,

(e) $\nu_1 \subseteq \nu_2 \Rightarrow f^{-1}(\nu_1) \subseteq f^{-1}(\nu_2)$ *for any fuzzy sets* ν_1, ν_2 *in* $f(S)$.

DEFINITION 1 ([1]). If $M = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ are additive abelian groups, and for all x, y, z in M and all α, β in Γ, the following conditions are satisfied

(1) $x\alpha y$ is an element of M,

(2) $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)y = x\alpha y + x\beta y$, $x\alpha(y+z) = x\alpha y + x\alpha z$,

(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

then M is called a Γ-ring.

In what follows, M and M' would mean Γ-rings unless otherwise specified.

DEFINITION 2 ([1]). A subset A of M is a left (right) ideal of M if A is an additive subgroup of M and

$$M\Gamma A = \{x\alpha y | x \in M, \alpha \in \Gamma, y \in A\}(A\Gamma M)$$

is contained in A. If A is both a left and a right ideal, then A is a two-sided ideal, or simply an ideal of M.

DEFINITION 3 ([2]). An ideal P of M is said to be prime if for every ideals A, B of M, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

PROPOSITION 1 ([2]). *Let P be an ideal of M. Then the following are equivalent:*

(a) *P is a prime ideal of M.*

(b) *For all $x, y \in M$, $x\Gamma M\Gamma y \subseteq P$ implies $x \in P$ or $y \in P$.*

DEFINITION 4 ([5]). A fuzzy set μ in M is called a fuzzy left (right) ideal of M if

(4) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$,

(5) $\mu(x\alpha y) \geq \mu(y) \quad (\mu(x\alpha y) \geq \mu(x))$,

for all $x, y \in M$ and all $\alpha \in \Gamma$.

A fuzzy set μ in M is called a fuzzy ideal of M if μ is both a fuzzy left and a fuzzy right ideal of M.

We note that μ is a fuzzy ideal of M if and only if

(4) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$,

(6) $\mu(x\alpha y) \geq \max\{\mu(x), \mu(y)\}$,

for all $x, y \in M$ and all $\alpha \in \Gamma$.

LEMMA 2 ([5]). *A fuzzy set μ in M is a fuzzy ideal of M if and only if the level subsets μ_t, $t \in Im(\mu)$, are ideals of M.*

REMARK 1. It follows from [5; Theorem 5] that the level ideals of a fuzzy ideal μ need not be distinct. Moreover, the level ideals form a chain. As $\mu(x) \leq \mu(0)$ for all $x \in M$, therefore the level ideal μ_t, $t = \mu(0)$, is smallest in the family of all level ideals of μ. If $Im(\mu) = \{t_0, t_1, ..., t_n\}$ with $t_0 > t_1 > ... > t_n$, then the chain of level ideals of μ is given by

$$\mu_{t_0} \subset \mu_{t_1} \subset ... \subset \mu_{t_n} = M.$$

DEFINITION 5 ([4]). Let μ and ν be fuzzy sets in M and let $\alpha \in \Gamma$. The product $\mu\Gamma\nu$ is defined by $\mu\Gamma\nu(x) = \sup\limits_{x=y\alpha z} \{\min\{\mu(y), \nu(z)\}\}$ and $\mu\Gamma\nu(x) = 0$ if x is not expressible as $x = y\alpha z$.

PROPOSITION 2 ([4]). *Let μ and ν be fuzzy left ideals of M. Then $\mu \cap \nu$ is a fuzzy left ideal of M (similar results hold for fuzzy right ideals and fuzzy ideals). If μ is a fuzzy right ideal and ν a fuzzy left ideal, then $\mu\Gamma\nu \subseteq \mu \cap \nu$.*

DEFINITION 6 ([4]). A fuzzy ideal μ of M is said to be prime if

(7) μ is not a constant function,

(8) for any fuzzy ideals ν, ρ in M, $\nu\Gamma\rho \subseteq \mu$ implies $\nu \subseteq \mu$ or $\rho \subseteq \mu$.

LEMMA 3 ([4]). *If μ is any nonempty fuzzy set in M, then μ is a fuzzy prime ideal of M if and only if $Im(\mu) = \{t_0, t_1\}$ where $t_0 = 1$ and $t_1 \in [0, 1)$, and the ideal $\mu_{t_0} = \{x \in M | \mu(x) = t_0 = 1\}$ is prime.*

DEFINITION 7. An ideal A of M is said to be irreducible if for any ideals I and J of M,

$$A = I \cap J \text{ implies } I = A \text{ or } J = A.$$

DEFINITION 8. A fuzzy ideal μ of M is said to be fuzzy irreducible if it is not an intersection of two fuzzy ideals of M properly containing μ.

THEOREM 1. *If μ is any fuzzy prime ideal of M, then μ is fuzzy irreducible.*

Proof. Assume that μ is not fuzzy irreducible. Then there exist fuzzy ideals ν and λ of M such that $\mu = \nu \cap \lambda, \mu \subset \nu$ and $\mu \subset \lambda$. From $\mu \subset \nu$ and $\mu \subset \lambda$, we have that $\mu(x) < \nu(x)$ and $\mu(y) < \lambda(y)$ for some $x, y \in M$. If μ is constant, then $\mu(x) = \mu(y) = \mu(x\alpha y)$ for all $\alpha \in \Gamma$. Since

$$\begin{aligned}(\nu\Gamma\lambda)(x\alpha y) &\geq \min\{\nu(x), \lambda(y)\}\\ &> \min\{\mu(x), \mu(y)\}\\ &= \mu(x\alpha y),\end{aligned}$$

it follows from Proposition 2 that $(\nu \cap \lambda)(x\alpha y) > \mu(x\alpha y)$. This is a contradiction. If μ is nonconstant, then by Lemma 3, $Im(\mu) = \{t_0, t_1\}$ where $t_0 = 1$ and $t_1 \in [0, 1)$; and the ideal $\mu_{t_0} = \{x \in M | \mu(x) = t_0 = 1\}$ is prime. Following Remark 1, we have that the chain of level ideals of μ is $\mu_{t_0} \subset M$. Thus there exists $s \in [0, 1)$ such that $\mu(x) = s$ for all $x \in M - \mu_{t_0}$. Since $\nu(x) > \mu(x)$ and $\lambda(y) > \mu(y)$, therefore $x, y \notin \mu_{t_0}$. From the fact that μ_{t_0} is a prime ideal of M, it follows that $x\Gamma M\Gamma y \not\subseteq \mu_{t_0}$, i.e., $x\alpha z\beta y \notin \mu_{t_0}$ for all $z \in M$ and all $\alpha, \beta \in \Gamma$, so that $\mu(x\alpha z\beta y) = s$. On the other hand

$$\begin{aligned}(\nu\Gamma\lambda)(x\alpha z\beta y) &\geq \min\{\nu(x\alpha z), \lambda(y)\}\\ &\geq \min\{\max\{\nu(x), \nu(z)\}, \lambda(y)\}\\ &= \min\{\nu(x), \lambda(y)\}\\ &> \min\{\mu(x), \mu(y)\}\\ &= s = \mu(x\alpha z\beta y).\end{aligned}$$

Hence $(\nu \cap \lambda)(x\alpha z\beta y) > \mu(x\alpha z\beta y)$, a contradiction. This completes the proof.

THEOREM 2. *Let μ be any non-constant fuzzy irreducible ideal of M. Then*

(a) $1 \in Im(\mu)$,

(b) *there exists $t \in [0,1)$ such that $\mu(x) = t$ for all $x \in M - \mu_{t_0}$, where $\mu_{t_0} = \{x \in M | \mu(x) = 1\}$, $t_0 = \mu(0) = 1$,*

(c) *the ideal μ_{t_0} is irreducible.*

Proof. (a). Assume that $1 \notin Im(\mu)$. Then $\mu(0) < 1$, say $\mu(0) = t_0$. Define fuzzy sets μ_1 and $\mu_2 : M \to [0,1]$ by

$$\mu_1(x) = \begin{cases} 1 & \text{if } x \in \mu_{t_0}, \\ \mu(x) & \text{otherwise} \end{cases}$$

and $\mu_2(x) = \mu(0)$ for all $x \in M$. It follows from Lemma 2 that μ_1 and μ_2 are fuzzy ideals of M. We now show that $\mu = \mu_1 \cap \mu_2$. If $x \in \mu_{t_0}$ then $\mu(x) \geq t_0 = \mu(0)$, and so $\mu(x) = \mu(0)$. But

$$\begin{aligned}(\mu_1 \cap \mu_2)(x) &= \min\{\mu_1(x), \mu_2(x)\} \\ &= \min\{1, \mu(0)\} \\ &= \mu(0) \\ &= \mu(x).\end{aligned}$$

If $x \in M - \mu_{t_0}$ then

$$(\mu_1 \cap \mu_2)(x) = \min\{\mu_1(x), \mu_2(x)\} = \min\{\mu(x), \mu(0)\} = \mu(x).$$

Hence $\mu = \mu_1 \cap \mu_2$. Clearly $\mu \subset \mu_1$ and $\mu \subset \mu_2$. This contradicts the fact that μ is fuzzy irreducible. Therefore $1 \in Im(\mu)$, so that $t_0 = \mu(0) = 1$.

(b). It is sufficient to show that the chain of level ideals of μ is precisely $\mu_{t_0} \subset M$. Let $\mu_t, t \in [0,1)$, be any level ideal of μ such that $\mu_{t_0} \subset \mu_t \subset M$. Then there exists $s_1 \in [0,t)$ such that

$$\mu(x) = \begin{cases} 1 & \text{if } x \in \mu_{t_0}, \\ t & \text{if } x \in \mu_t - \mu_{t_0}, \\ s_1 & \text{if } x \in M - \mu_t. \end{cases}$$

Define fuzzy sets μ_3 and $\mu_4 : M \to [0,1]$ as follows:

$$\mu_3(x) = \begin{cases} 1 & \text{if } x \in \mu_t, \\ s_1 & \text{otherwise}, \end{cases} \qquad \mu_4(x) = \begin{cases} 1 & \text{if } x \in \mu_{t_0}, \\ t & \text{if } x \in \mu_t - \mu_{t_0}, \\ s_2 & \text{if } x \in M - \mu_t, \end{cases}$$

where $s_1 < s_2 < t$. By routine calculations, we know that μ_3 and μ_4 are fuzzy ideals of M. Next we show that $\mu = \mu_3 \cap \mu_4$. If $x \in \mu_{t_0}$ then

$$(\mu_3 \cap \mu_4)(x) = \min\{\mu_3(x), \mu_4(x)\} = 1 = \mu(x).$$

If $x \in \mu_t - \mu_{t_0}$ then

$$(\mu_3 \cap \mu_4)(x) = \min\{\mu_3(x), \mu_4(x)\} = t = \mu(x).$$

If $x \in M - \mu_t$ then

$$(\mu_3 \cap \mu_4)(x) = \min\{\mu_3(x), \mu_4(x)\} = s_1 = \mu(x).$$

Thus $\mu = \mu_3 \cap \mu_4$. It is clear that $\mu \subset \mu_3$ and $\mu \subset \mu_4$. This contradicts the fact that μ is fuzzy irreducible. Hence the chain of level ideals of μ is $\mu_{t_0} \subset M$, so that

$$\mu(x) = \begin{cases} 1 & \text{if } x \in \mu_{t_0}, \\ s & \text{if } x \in M - \mu_{t_0}, \end{cases}$$

for some $s \in [0, 1)$.

(c). Assume that μ_{t_0} is not irreducible. Then there exist ideals A and B of M such that $\mu_{t_0} = A \cap B, \mu_{t_0} \subset A, \mu_{t_0} \subset B$. Thus A is not contained in B and B is not contained in A, and $(A - \mu_{t_0}) \cap (B - \mu_{t_0})$ is the empty set. Let μ_5 and μ_6 be fuzzy sets in M defined by

$$\mu_5(x) = \begin{cases} 1 & \text{if } x \in \mu_{t_0}, \\ t_0' & \text{if } x \in A - \mu_{t_0}, \\ s & \text{if } x \in M - A, \end{cases} \qquad \mu_6(x) = \begin{cases} 1 & \text{if } x \in \mu_{t_0}, \\ t_0' & \text{if } x \in B - \mu_{t_0}, \\ s & \text{if } x \in M - B, \end{cases}$$

where $s < t_0' < 1$. Then μ_5 and μ_6 are fuzzy ideals of M satisfying

$$\mu = \mu_5 \cap \mu_6, \mu \subset \mu_5, \mu \subset \mu_6.$$

This is impossible as μ is fuzzy irreducible. The proof is complete.

DEFINITION 9. A mapping $f : M \to M'$ is called a Γ-homomorphism if $f(x+y) = f(x) + f(y)$ and $f(x\alpha y) = f(x)\alpha f(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

LEMMA 4 ([5]). *(a) A Γ-homomorphic preimage of a fuzzy left (right) ideal is a fuzzy left(right) ideal. (b) A Γ-homomorphic image of a fuzzy left(right) ideal which has sup property is a fuzzy left(right) ideal.*

THEOREM 3. *Let $f : M \to M'$ be a Γ-homomorphism. Then*

(a) if μ is a f-invariant fuzzy irreducible ideal of M with sup property, then $f(\mu)$ is a fuzzy irreducible ideal of M'.

(b) if μ' is any fuzzy irreducible ideal of M' and if every fuzzy ideal of M is f-invariant, then $f^{-1}(\mu')$ is a fuzzy irreducible ideal of M.

Proof. By Lemma 4, $f(\mu)$ and $f^{-1}(\mu)$ are fuzzy ideals of M' and M respectively. Assume that $f(\mu)$ is not fuzzy irreducible. Then there exist fuzzy ideals ν_1 and ν_2 of M' such that $f(\mu) = \nu_1 \cap \nu_2, f(\mu) \subset \nu_1, f(\mu) \subset \nu_2$. As μ is f-invariant, it follows from Lemma 1 that

$$\mu = f^{-1}(\nu_1 \cap \nu_2),\ \mu \subset f^{-1}(\nu_1) \text{ and } \mu \subset f^{-1}(\nu_2).$$

To show that $f^{-1}(\nu_1 \cap \nu_2) = f^{-1}(\nu_1) \cap f^{-1}(\nu_2)$, let x be any element of M. Then

$$\begin{aligned}(f^{-1}(\nu_1 \cap \nu_2))(x) &= (\nu_1 \cap \nu_2)(f(x)) \\ &= \min\{\nu_1(f(x)), \nu_2(f(x))\} \\ &= \min\{(f^{-1}(\nu_1))(x), (f^{-1}(\nu_2))(x)\} \\ &= (f^{-1}(\nu_1) \cap f^{-1}(\nu_2))(x),\end{aligned}$$

which implies that

$$f^{-1}(\nu_1 \cap \nu_2) = f^{-1}(\nu_1) \cap f^{-1}(\nu_2).$$

Hence $\mu = f^{-1}(\nu_1) \cap f^{-1}(\nu_2), \mu \subset f^{-1}(\nu_1), \mu \subset f^{-1}(\nu_2)$, which contradicts the fact that μ is fuzzy irreducible. Therefore $f(\mu)$ is fuzzy irreducible, which proves (a).

To prove (b), assume that $f^{-1}(\mu')$ is not fuzzy irreducible. Then $f^{-1}(\mu') = \sigma_1 \cap \sigma_2$, $f^{-1}(\mu') \subset \sigma_1$ and $f^{-1}(\mu') \subset \sigma_2$ for some fuzzy ideals σ_1 and σ_2 of M. It is evident from Lemma 1 that $\mu' = f(\sigma_1 \cap \sigma_2)$, $\mu' \subset f(\sigma_1)$ and $\mu' \subset f(\sigma_2)$. Now we show that $f(\sigma_1 \cap \sigma_2) = f(\sigma_1) \cap f(\sigma_2)$. Since $\sigma_1 \cap \sigma_2 \subseteq \sigma_1$ and $\sigma_1 \cap \sigma_2 \subseteq \sigma_2$, it follows from Lemma 1(c) that $f(\sigma_1 \cap \sigma_2) \subseteq f(\sigma_1) \cap f(\sigma_2)$. To establish the reverse inclusion, let $y \in M'$, $t = (f(\sigma_1) \cap f(\sigma_2))(y)$ and $\epsilon > 0$ be any real number. Then

$$\begin{aligned} t - \epsilon &< \min\{(f(\sigma_1))(y), (f(\sigma_2))(y)\} \\ &= \min\{\sup_{x \in f^{-1}(y)} \sigma_1(x), (f(\sigma_2))(y)\}, \end{aligned}$$

which implies that $t - \epsilon < \sigma_1(z)$ for some $z \in f^{-1}(y)$ and $t - \epsilon < (f(\sigma_2))(y)$. This means that $t - \epsilon < \sigma_1(z)$ and

$$\begin{aligned} t - \epsilon &< (f(\sigma_2))(f(z)) \\ &= (f^{-1}(f(\sigma_2)))(z) \\ &= \sigma_2(z) \quad \text{since } \sigma_2 \text{ is } f\text{-invariant.} \end{aligned}$$

Hence

$$t - \epsilon < \min\{\sigma_1(z), \sigma_2(z)\} = (\sigma_1 \cap \sigma_2)(z).$$

From $z \in f^{-1}(y)$ it follows that

$$t - \epsilon < \sup_{x \in f^{-1}(y)} (\sigma_1 \cap \sigma_2)(x) = (f(\sigma_1 \cap \sigma_2))(y).$$

As $\epsilon > 0$ was arbitrary, therefore

$$t = (f(\sigma_1) \cap f(\sigma_2))(y) \leq (f(\sigma_1 \cap \sigma_2))(y),$$

so that

$$f(\sigma_1) \cap f(\sigma_2) \subseteq f(\sigma_1 \cap \sigma_2).$$

Hence $\mu' = f(\sigma_1) \cap f(\sigma_2)$, $\mu' \subset f(\sigma_1)$ and $\mu' \subset f(\sigma_2)$, which contradicts the fact that μ' is fuzzy irreducible, The proof is complete.

The following theorem is an immediate consequence of Lemma 1 and Theorem 3.

THEOREM 4. *Let $f : M \to M'$ be a Γ-homomorphism and let every fuzzy ideal of M be f-invariant. Then the mapping $\sigma \mapsto f(\sigma)$ defines a 1-1 correspondence between the set of all fuzzy irreducible ideals of M with sup property and the set of all fuzzy irreducible ideals of M'.*

References

1. W. E Barnes, *On the Γ-rings of Nobusawa*, Pacific J. Math. **18** (1966), 411 - 422
2. G. L Booth and N J Groenwald, *On uniformly strongly prime gamma rings*, Bull Austral Math Soc. **37** (1988), 437 - 445
3. P S Das, *Fuzzy groups and level subgroups*, J Math Anal Appl **84** (1981), 264 - 269
4. Y B Jun, *On fuzzy prime ideals of Γ-rings*, (submitted)
5. Y. B. Jun and C Y Lee, *Fuzzy Γ-rings*, Pusan Kyongnam Math J. **8** (1992), 163 - 170
6. R. Kumar, *Fuzzy irreducible ideals in rings*, Fuzzy Sets and Systems **42** (1991), 369 - 379
7. T. K Mukherjee and M. K Sen, *On fuzzy ideals of a ring I*, Fuzzy Sets and Systems **21** (1987), 99 - 104.
8. T. S Ravisankar and U S. Shukla, *Structure of Γ-rings*, Pacific J. Math **80** (1979), 537 - 559
9. A. Rosenfeld, *Fuzzy groups*, J Math Anal Appl **35** (1971), 512 - 517.
10. U. M Swamy and K L. N. Swamy, *Fuzzy prime ideals of rings*, J Math. Anal Appl. **134** (1988), 94 - 103
11. L A Zadeh, *Fuzzy sets*, Inform Control **8** (1965), 338 - 353

Department of Mathematics Education
Gyeongsang National University
Chinju 660-701, Korea
Fax: To Prof. Y. B. Jun, 82-591-751-6117

Department of Mathematics Education
Kyungnam University
Masan 631-701, Korea