Pusan Kyŏngnam Math J 9(1993), No. 2, pp. 241–249

ON EPIC AND MONIC ENDOMORPHISMS

BAE, SOON-SOOK

0. Introduction

Assume that ring R is an associtive ring with an identity and every module $M = {}_RM$ is a left R-module.

The ring of all R-endomorphisms on a left R-module M, denoted by $End(M)$ will be written on the right side of M as right operators on M, that is, ${}_RM_{End(M)}$ will be considered on this paper. For R-homomorphisms $f : L \to M$, $g : M \to N$, their composition fg : $L \xrightarrow{f} M \xrightarrow{g} N$ of f and g is written in the arrow orders of f and g, for any left R-modules L, M and N.

According to [5], a module ${}_RM$ is *quasi-projective* in case it is $M-$ *projective* (or,*projective* relative to M itself). This definition is equivalent to the following definition of [1](see 16.7 Proposition p148,[5]). In other words, we can replace N with M/T for each submodule T of M.

DEFINITION 1. An R-module M is said to be *quasi-projective* if every diagram

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow{\scriptstyle f} & & \\ M & \xrightarrow{\ g\ } & N & \longrightarrow & 0 \end{array}$$

where the bottom row is exact with f,g are $R-$homomorphisms, has a commutative diagram;

$$\begin{array}{ccccc} M & = = = & M & & \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} & & \\ M & \xrightarrow{\ g\ } & N & \longrightarrow & 0 \end{array}$$

there exists an endomrophism h on M such that $f = hg$.

Dedicated to Professor Younki Chae on his 61st birthday

Received September 9,1993 .

THEOREM 2. [3] *Let M be a (quasi-)projective left R-module. Then $Rad(End(M)) = \{f \in End(M) | Imf$ is superfluous (or small) in $M\}$.*

Let's use the notation $E(M)$ to stand for an *injective hull* of M.

THEOREM 3. [6, p49] *Let M be a (quasi-)injective left R-module. Then,*
$Rad(End(M)) = \{f \in End(M) | \ kerf$ is essential (or large) in $M\}$.

DEFINITION 4. ([6], p 48) A module M is said to be *quasi-injective* provided the natural map $Hom_R(M, M) \to Hom_R(L, M)$ is a surjective for all $L \leq M$, i.e., provided any homomorphism from a submodule of M into M extends to an endomorphism of M.

1. Results

In order to reveal the relationships between submodules of a module and ideals of its endomorphism rings, firstly we must find the structures of the radical of the endomorphism whose element is left-(or right)*quasi-regular* .To do this, we need to see whether endomorphisms are left invertible or right invertible, or not. Notice the composition of maps follows arrow- direction. To make a comparison of *(quasi-)projectivity* with *(quasi-)injectivity*, I will write some results from [10] with respect to *(quasi-)injectivity*.

THEOREM 1. *Let a left R-module $_RM$, be (quasi-)projective and let f be an endomorphism in $End(_RM)$. Then we have f is an epimorphism if, and only if , f has a left inverse.*

Proof. " If " part is easy. Now let's prove the " only if " part. Let $f : M \to M$ be an epimorphism. Then we have the induced isomorphism $\overline{f} : M/kerf \to M$ by the first isomorphism theorem. Consider the following diagram with the natural map n;

$$\begin{array}{ccccc} M & = & M & & \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \overline{f}^{-1}} & & \\ M & \xrightarrow{n} & M/kerf & \longrightarrow & 0 \end{array}$$

and then we have an $R-$endomorphism $h : M \to M$ such that $\overline{f}^{-1} = hn$, from which we also have $hn\overline{f} = 1_M$. Actually $hn\overline{f} = hf = 1_M$ which proves the theorem.

THEOREM 2. *For a left (quasi-)injective R-module $_RM$, and an endomorphism f, f is a monomorphism if, and only if, f has a right inverse.*

Proof. " If " part is easy. Now let's prove the " only if " part. Since every module $_RM$ has an *injective envelope* $E(_RM) = E(M)$, we need to show that each monomorphism $f : 0 \to M \xrightarrow{f} M$, there exists an R-module homomorphism $g : M \to M$ such that $fg = 1_M$. From the the definition of injective envelope $E(M)$, for a monomorphism $fi : M \to E(M)$, there is an extension $\overline{f} : E(M) \to E(M)$ such that $\overline{f}|_M = fi$. Now that there exists a $\overline{g} : E(M) \to E(M)$ such that $\overline{f}\overline{g} = 1_{E(M)}$, $Im\,\overline{f}$ is a direct summand of $E(M)$.

Let's put $g = \overline{g}|_M = ig : M \subset E(M) \to E(M)$, then for each $x \in M$, $xfg = xfig = x\overline{f}g = x\overline{f}\overline{g} = x1_{E(M)} = x$. Thus $fg = 1_M$.

REMARK 3. *The condition "(quasi-)projectivity" of M in the above theorem 1 is necessary for any epimorphism having a left inverse in the endomorphism ring of M. For an example, take $R = Z$, $_RM = Z(p^\infty)$, the multiplication by p,in fact,it is an epimorpism in End($_RM$). We now have an epimorphism having no left inverse in End(M). Without any hesitation we have that $Z(p^\infty)$ is not a (quasi-)projective $Z-$module.*

In the above theorem 2, there is no guarantee for existing the inverse g of a monomorphism f in End(M). For an example, take $R = Z$, $_RM = Z$, $f = \times 2$, the multiplication by 2, then Z has its envelope Q, f has its right inverse $g = \div 2$ the division by 2 whose range $\{\frac{1}{2}s \mid s \in Z\}$ is not contained in $Z = {_RM}$. However f has g as its inverse homomorphism in $Hom_R(Imf, Z)$ or $Hom_R(Z, Q)$. Immediately, we conclude that Z is not (quasi-)injective .

Here we can add one more equivalent condition for an endomorphism f to be an epimorphism to the Proposition 3.4. p44 in [5].

PROPOSITION 4. *For any (quasi-)projective left $R-$module M and $f : M \to M$ an endomorphism the followings are equivalent :*

(a) *f is an epimorphism .*

(b) *Im*$(f) = M$.
(c) *For every* $_RK$ *and every pair* $g, h : M \to K$ *of* R *- homomorphisms,* $fg = fh$ *implies* $g = h$.
(d) *For every* $_RK$ *and every R-homomorphism* $g : M \to K$, $fg = 0$ *implies* $g = 0$.
(e) f *has a left inverse ,in fact, in* $End_R(M)$.

Proof. Proof is immediately followed from Theorem 1.

Similarly, we can add one more equivalent condition of an endomorphism f to be a monomorphism to the proposition 3.4. p 44 in [5]

PROPOSITION 5. *For any left (quasi-)injectiveR-*module M *and an endomorphism* $f : M \to M$, *the followings are equivalent :*

(a) f *is a monomorphism* .
(b) $\ker f = \{0\}$.
(c) *For every* $_RK$ *and every pair* $g, h : K \to M$ *of* R *- homomorphisms,* $gf = hf$ *implies* $g = h$.
(d) *For every* $_RK$ *and every R-homomorphism* $g : K \to M$, $gf = 0$ *implies* $g = 0$.
(e) f *has a right inverse. (In fact, in* $Hom_R(Imf, M)$ *or* Hom_R $(M, E(M))$, *where* $E(M)$ *is an injective envelope of* M.*)*

Proof. Proof is immediately followed from Theorem 2.

The following corollaries are followed easily, and thus we omit those proofs.

COROLLARY 6. *In the endomorphism ring* $End(_RM)$ *of any (quasi -)projective left R-module* $_RM$, *if the composition* gf *is an epimorphism with* $f, g \in End(_RM)$, *then so is* f.

COROLLARY 7. *In the endomorphism ring* $End(_RM)$ *of any left (quasi-)injective R-module* $_RM$, *if the composition* fg *is a monomorphism, so is* f.

LEMMA 8. *If* $_RM$ *is (quasi-)projective. Then no epimorphism in* $End(_RM)$ *is contained in a left proper ideal of* $End(_RM)$.

Proof. Suppose that a proper left ideal I of $End(_RM)$ contains an epimorphism f. Since $_RM$ is *(quasi-)projective*, by theorem 1, there exists an endomorphism g in $End({}_RM)$ such that $gf = 1_M$ which implies that $I = End({}_RM)$. It contradicts to be a proper left ideal I.Hence we completes the proof.

LEMMA 9. *If ${}_RM$ is (quasi-)injective. Then no monomorphism in $End({}_RM)$ is contained in a right proper ideal of $End({}_RN)$.*

Proof. Proof is easily followed by the same way of the above corollary.

Now, it is worth considering left ideals of $End(M)$ of the form

$$I^N = Hom_R(M, N) = \{f \in End(M) \mid Imf \subseteq N\},$$

for a submodule N of M. and right ideals of $End(M)$ of the form

$$I_N = \{f \in End(M) \mid N \subseteq ker\, f\},$$

for a submodule N of M.

THEOREM 10. *For a (quasi-)projective module ${}_RM$ and any small(or superfluous) submodule N of M, the right ideal I^N is small in $End(M)$.*

Proof. We need only consider all left ideals of $End(M)$. Suppose that J is a left ideal of $End(M)$ such that $I^N + J = End(M)$. Then the identity 1_M can be written as a sum of $f \in I^N$ and $j \in J$, i.e., $1_M = f + j$. Then $M = Im1_M = Im(f+j) \leq Imf + Imj \leq N + Imj$, which implies that $N + Imj = M$. Since N is small in M, we have $Imj = M$, saying that j is an epimorphism. By Theorem 1, j has a left inverse in $End(M)$, hence $J = End(M)$. Therefore we have proved I^N is small in $End(M)$.

THEOREM 11. *For a (quasi-)injective module ${}_RM$ and any large submodule N of M, the right ideal I_N is small in $End(M)$.*

Proof. We need only consider all right ideals of $End(M)$. Suppose that J is a right ideal of $End(M)$ such that $I_N + J = End(M)$. Then the identity 1_M can be written as a sum of $f \in I_N$ and $j \in J$, i.e., $1_M = f + j$. Then $0 = ker1 = ker(f+j) \geq kerf \cap kerj \geq N \cap kerj$, which implies that $N \cap kerj = 0$. Since N is large in M, we have $kerj = 0$, saying that j is a monomorphism. By Theorem 2, j has a right inverse in $End(M)$, hence $J = End(M)$. Therefore we have proved I_N is small in $End(M)$.

Here is an easier proof than that had done in [1] and a similar method of proof in [5] of the following fact restated Lemma 1 in [3] by using our tool I^N.

LEMMA 12. ([3]) *Let M be a (quasi-)projective left R-module. Then,*

$$Rad(End(M)) = \{f \in End(M) | Imf \text{ is superfluous (or small) in } M\}$$

Proof. Suppose that an endomorphism f has a small image Imf in M. Considering the left ideal $I^{Imf} = \{h \in End(M) | Imh \subseteq Imf\}$, we have a small left ideal I^{Imf} of $End(M)$ by Theorem 6,which contains f. Since the radical $Rad(End(M))$ is the largest small left ideal of $End(M)$,$f \in I^{Imf} \subseteq Rad(End(A))$. Conversely let f be in $Rad(End(M))$ and let K be a submodule of M such that $Imf + K = M$.

$$\begin{array}{ccccccc} & & & & M & & \\ & & g & & \downarrow n_K & & \\ M & \xrightarrow{f} & M & \xrightarrow{n_K} & M/K & \longrightarrow & 0 \end{array}$$

where n_K is the natural epimorphism. Since fn_K is an epimorphism, there exists an endomorphism $g : M \to M$ such that $gfn_K = n_K$. Thus we have $(1 - gf)n_K = 0$. Since $f \in Rad(End(M)$, $1 - gf$ is invertible, and so $n_K = 0$. Hence $K = M$ which completes the proof.

REMARK 13. According to the propositions ([4],p118) and [8] ,the radical $Rad(M)$, the socle $Soc(M)$ of M and the relations are;

$$\begin{aligned} Rad\, M &= \bigcap \{K < M | K \text{ is maximal in} M\} \\ &= \sum \{L < M | L \text{ is small in } M\}, \end{aligned}$$

$$\textstyle\sum_{a \in A} I^{N_a} \leq I^{\sum_{a \in A} N_a} \text{ for every } N_a \leq M,$$

$$Soc\,M = \sum\{K < M \mid K \text{ is } minimal \text{ in } M\}$$
$$= \bigcap\{L < M \mid L \text{ is } large \text{ in } M\},$$
$$\text{and} \quad \sum_{a\in A} I_{N_a} \leq I_{\cap_{a\in A} N_a} \text{ for every } N_a \leq M.$$

THEOREM 14. *For any left (quasi-)projective module M with $Rad\,M = \sum_{a\in A} N_a$, small submodule N_a of M, we have*

$$\sum_{a\in A} I^{N_a} = Rad(End(M)).$$

Proof. Since $Rad(End(M))$ is the unique largest small left ideal of $End(M)$, thus for every small submodule N_a of M, the small left ideal I^{N_a} is contained in the radical $Rad(End(M))$ of $End(M)$ for every $a \in A$. Hence the sum $\sum_{a\in A} I^{N_a}$ is also contained in $Rad(End(M))$. Conversely, assume f is in $Rad(End(M))$, Imf is smalll in M by Theorem 8. And I^{Imf} is a small left ideal of $End(M)$, $f \in I^{Imf} \leq \sum_{a\in A} I^{N_a}$ for every small submodule N_a of M. Thus we complete the proof.

THEOREM 15. *For any left (quasi-)injective module M with $Soc\,M = \bigcap_{a\in A} N_a$, large submodule N_a of M, we have*

$$\sum_{a\in A} I_{N_a} = Rad(End(M)).$$

Proof. From Theorem in [5], $Rad(End(M))$ is the unique largest *small* right ideal of $End(M)$, and thus for every *large* submodule N_a of M, the *small* right ideal I_{N_a} is contained in the radical $Rad(End(M))$ of $End(M)$ for every $a \in A$. Hence the sum $\sum_{a\in A} I_{N_a}$ is also contained in $Rad(End(M))$.

Conversely, assume f is in $Rad(End(M))$, $ker f$ is *large* in M by Theorem 3 in the introduction. And I_{kerf} is a *small* right ideal of $End(M)$, $f \in I_{kerf} \leq \sum_{a\in A} I_{N_a}$ for every *large* submodule N_a of M. Thus we complete the proof.

REMARK 16. We summarize these two theorems 14,15, then we conclude;

(1) In a *(quasi-)projective* module M,if the radical $Rad(M)$ of M is small,
easily we can conclude that $Rad(End(M)) = I^{Rad(M)}$.
(2) In a *(quasi-)injective* module M with large socle $soc(M)$,
we obtain $Rad(End(M)) = I_{soc(M)}$.
(3) Moreover in a left *(quasi-)projective* and *(quasi-) injective* module M with (1) ,or (2) condition, we have $Rad(End(M) = I^{Rad(M)}$,or $I_{soc(M)}$, respectively.

The following corollaries are followed immediately.

COROLLARY 17. *For any left (quasi-)projective module M, we have the followings :*

(a) *If $Rad(End(M)) = 0$, then there is no non-zero endomorphism whose image is small in M ;*
(b) $Rad\,(End(M)) \leq I^{Rad(M)}$;
(c) *If $Rad(M) = 0$, then $Rad\,(End(M)) = 0$.*

COROLLARY 18. *For any left (quasi-)injective module M, we have the followings :*

(a) *If $Rad(End(M)) = 0$, then there is no non-zero endomorphism whose kernel is large in M ;*
(b) $Rad\,(End(M)) \leq I_{Soc(M)}$;
(c) *If $Soc(M) = M$, then $Rad\,(End(M)) = 0$.*

EXAMPLES 19. Since it is well-known that $M = Z(2^\infty)$ is injective, and $\{\overline{0}, \overline{(1/2)}\}$ is the socle of M, and it is *essential (i.e., large)* in M, and thus $End(M)$ has the radical $I_{soc(M)}$ isomorphic to $2Z$.

In the second example, when $R = Z = M$, $M = Z$ has zero radical. Since M is a *free, (quasi-) projective* module with $Rad(M) = 0$, its endomorphism ring $End(M)$ has radical $I^{Rad(M)} = I^0 = 0$.

References

1. L.E.T. Wu and J.P. Jans, *On Quasi Projectives*, Illinois J.Math **11** (1967), 439–448.
2. I.N.Herstein, *Noncommutative rings*, The Math. Assoc. of America, 1968.

3. R. Ware and J. Zelmanowitz, *The Jacobson Radical of the Endomorphism Ring of a projective module*, Proc. AMS **26** (1970), 15–20.
4 Roser Ware, *Endomorphism Rings of Projective Modules*, Trans of the AMS **55** (1971), 233–256
5. Frank W Anderson and Kent R.Fuller, *Rings and Categories of Modules*, New York , Spring -Verlag, 1973
6 K.R Goodearl, *Ring Theory , Nonsingular Rings and Modules*, Marcel Dekker, Inc , 1976.
7. Joseph J Rotman, *An introduction to Homological Algebra*, Academic Press, 1979.
8. Bae, Soon-Sook, *On Ideals of Endomorphism Rings*, Dr of Sc Thesis, Kyung-pook Univ June, 1989.
9. Dinh Van Huynh and Robert Wisbauer, *A Structure Theorem for SI-Modules*, Glasgow Math.J. **34** (1992), 83–89.
10 Bae S.S.,Kim J M and Jung Y.S , *On Monomorphisms in the Endomorphism Ring*, The Jr of the Inst.of Natural Science,Kyungnam Univ. **5** (1993), 5–9.

Department of Mathematics
Kyungnam University
Masan, 630-701 Korea