Pusan Kyŏngnam Math. J 9(1993), No. 2, pp 235–239

A STUDY ON RATIO OF THE MAXIMUM TO THE MINIMUM OF TWO NORMAL SAMPLE VARIANCE

CHOON-IL PARK

1. INTRODUCTION

Consider the analysis of variance models, where for testing the equality of different effects of a factor we are dealing with the distribution of the ratio of independent (noncentral) chi squared random variables (r.v.'s). Thus, instead of considering the ratio of two independent chi squared r.v.'s as is the case in Pandey and Bhattacharya(1986), we consider the following more general setup. Let W_1 and W_2 be independent r.v.'s with Wi having a noncentral chi squared distribution with v_i degrees of freedom(df) and noncentrality parameter λ_i $(i = 1,2)$. Let

$$T_2 = \max\left[\frac{W_1}{v_1}, \frac{W_2}{v_2}\right],\ T_1 = \min\left[\frac{W_1}{v_1}, \frac{W_2}{v_2}\right],\ and\ W = \frac{T_2}{T_1}.$$

2. DISTRIBUTION OF W

Let X_1 and X_2 be independent positive r.v.'s with densities f_1 and f_2, respectively. Define $M = \max\{X_1, X_2\}$, $m = \min\{X_1, X_2\}$, and $V = M/m$ $(1 \le V < \infty,\ 0 < m < \infty)$. Then, it can be easily seen that the density function of V is of form:

$$f_V(v) = \int_R [f_1(u)f_2(uv) + f_1(uv)f_2(u)]u\,du\ (1 \le v < \infty). \qquad (1)$$

Received in September, 1993 .

Let $f_{v_1,v_2}(x)$ denote the density of a central F–distribution with the numerator df v_1 and the denominator df v_2. We introduce the following "modified" noncentral F–distribution with densities:

$$f^{(1)}_{v_1,v_2;\lambda}(x) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} f_{(v_1+2k),v_2}(x)\ (0 < x < \infty) \tag{2}$$

$$f^{(2)}_{v_1,v_2;\lambda}(x) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} f_{v_1,(v_2+2k)}(x) \tag{3}$$

Then using (1), (3), the main result is given bg the following theorem.

Theorem. The density function of W is given by

$$\begin{aligned} f_W(w) &= \sum_{k_1=0}^{\infty} \frac{e^{-\lambda_1}\lambda_1^{k_1}}{k_1!} f_{(v_1+2k_1),v_2;\lambda_2}(w) \\ &\quad + \sum_{k_2=0}^{\infty} \frac{e^{-\lambda_2}\lambda_2^{k_2}}{k_2!} f_{(v_2+2k_2),v_1;\lambda_1}(w)\ (1 \le w < \infty) \\ &= \sum_{k_2=0}^{\infty} \frac{e^{-\lambda_2}\lambda_2^{k_2}}{k_2!} f_{v_1,(v_2+2k_2);\lambda_1}(w) \\ &\quad + \sum_{k_1=0}^{\infty} \frac{e^{-\lambda_1}\lambda_1^{k_1}}{k_1!} f_{v_2,(v_1+2k_1);\lambda_2}(w),\ (1 \le w < \infty), \end{aligned}$$

where $f^{(1)}$ and $f^{(2)}$ are defined in (2) and (3), respectively.

Clearly, if W_1 and W_2 are (central) chi squared r.v.'s (i.e., $\lambda_1 = \lambda_2 = 0$), then, from theorem, the density function of W is iven by

$$f_W(w) = f_{v_1,v_2}(w) + f_{v_2,v_1}(w)\ (1 \le w < \infty)$$

which is equation of Pandey and Bhattacharya (1986).

As an application of the proposed test, consider the one-way fixed effect model:

$$y_{ij} = \mu + \alpha_i + e_{ij}\ (i = 1, \ldots,\ a_{ij} = 1, \ldots, n),$$

where μ is the general mean effect, α_i's are treatment effects with $\sum_{i=0}^{a} \alpha_i = 0$, and e_{ij}'s are i.i.d. $N(0, \alpha_e^2)$ r.v.'s. Define

$$MS_a = n \sum_{i=1}^{a} \frac{\overline{y}_{i.} - \overline{y}_{..})^2}{a-1} \; and \; MS_r = \sum_{i=1}^{a} \sum_{j=1}^{n} \frac{(y_{ij} - \overline{y}_{i.})^2}{a(n-1)},$$

where

$$\overline{y}_i = \frac{y_{ij}}{n} \; and \; \overline{y} = \sum_{i=1}^{a} \sum_{j=1}^{n} \frac{y_{ij}}{an}.$$

MS_a is the treatment mean sum of squares and MS_r is the residual mean sum of squares. Under $H_0 : \alpha_1 = \cdots = \alpha_a$ the distribution of $(a-1)MS_a/\sigma_e^2$ is a chi squared with $(a-1)$ df, where as under $H_1 : \alpha_i \neq \alpha_{i'}$, for at least a pair of indices (i, i'), $i \neq i' = 1, \cdots, a$, the distribution of $(a-1)MS_a/\sigma_e^2$ is a noncentral chi squared with $(a-1)$ df and noncentrality parameter $\sum_{i=1}^{a} \frac{\alpha_i^2}{2\sigma_e^2}$. The alternative and null distributions of $a(n-1)MS_r/\sigma_e^2$ are chi squared wiht $a(n-1)$df. The alternative and null distributions of $W = \max\{MS_a, MS_r\}$ are given by the density functions.

$$fw(w) = \sum_{k_i=0}^{\infty} \frac{e^{-\lambda_1} \lambda_1^{k_1}}{k_1!} f_{(v_1+2k_1),v_2}(w) + \sum_{k_1=0}^{\infty} \frac{e^{-\lambda_1} \lambda_1^{k_1}}{k_1!} f_{v_2,(v_1+2k_1)}(w) \quad (1 \leq w < \infty)$$

and

$$fw(w) = f_{v_1,v_2}(w) + f_{v_2,v_1}(w) \quad (1 \leq w < \infty)$$

respectively, where $v_1 = (a-1)$ and $v_2 = a(n-1)$.

For the data summarized in ANOVA 5.4 of Dunn and Clark (1987). $MS_a = 1.23$, $MS_r = 0.34$, $a = 4$, $n = 6$. The value of W test is $W = \frac{1.23}{0.34} = 3.62$ which is also the value of the usual F–test statistic MS_a/MS_r. The P-value (that is the observed level of significance) for the W–test is approximately 0.10 where the P–value for the F–test is between 0.025 and 0.05. This shows that W–test is a very conservative test.

P.C. Tang has compiled tables that can be used to compute the power function of a noncentral F–distribution (see, e.g., Graybill, 1961, vol.1, pp.444–459 for Tang's tables). The computations were done using a computer program that has several subruotines from Press, Fannery, Teukolsky and Vetterling (1988). The power of the noncentral F–distribution with $v_1 = 5$, $v_2 = 12$, $\lambda_1 = 0(1)10$ and for $\alpha = 0.05$, 0.10 are given in brackets in the row corresponding to $\lambda_2 = 0$. These values coincide with Tang's table. Tables show that the proposed test has very small power compared to the usual F–test. Similar to W test comparing estimated spectra densities. Coates and Diggle (1986) considered the ratio U/L, where $U = \max_{1 \leq j \leq m} R(w_j)$, $L = \min_{1 \leq j \leq m} R(w_j)$'s are the scaled(central) F–ratios. They also observed that the test U/L is not particularly powerful.

In conclusion, we fell that the W–test is not an appropriate alternative test to be used in practice. Thus, even in the two sample problem of testing the equality of variances, one should the usual F–ratio test with the larger of the sample variances in the numerator, and pretend as if this is indeed the real situation.

REFERENCES

1. D.S. COATES, P.J. DIGGLE (1986), *Tests for comparing two estimated spectral densities*, "Journal of Time Series Analysis", 7, pp.7–20.
2. O.J. DUNN, V.A. CLARK (1987), *Applied statistics: Analysis of variance and regression*, (second edition), John Wiley, New York.
3. F.A. GRAYBILL (1961), *An introduction to linear statistical models*, vol. 1, McGraw-Hill, New York.
4. J.T. McCLAVE, F.H. DIETRICH, II (1988), *Statistics*, Dellen Publishing Co., San Francisco.
5. G.S. PANDEY, S.K. BHATTACHARYA (1986), *Ratio of the maximum to the minimum of three normal sample variances*, "Statistica", 3, pp.537–539.

6. W.H. PRESS, B.P. FLANNERY, S.A. TEUKOLSKY, W.T. VETTERLING (1988), *Numerical recipes in C: the art of scientific computing*, Cambridge University Press, Cambridge.

Department of Applied Mathmetics
Korea Maritime University
Pusan, 606-791, Korea