投資收益이 保險需要에 미치는 影響에 관한 理論的 考察

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〈요　　약〉

　　보험업자는 보험계약자가 사전에 納付한 보험료를 사채·주식등의 유가증권에 투자하여 보험본연의 업무에서 보다 더 큰 수익을 얻고 있다. 한편, 보험수요자도 보험을 구입함과 동시에 효율적인 富의 운용을 위해 투자에 참여하는 것이 일반적이다. 이런 관점에서 본 논문은 전통적인 보험수요모형에 채권과 수익에의 투자결정을 内在化함으로써 투자수익이 고려된 균형보험료의 재무경제학적인 意味를 규명하고 전통적인 보험이론을 수정·보완하는 데 그 의의가 있다. 본 논문에서 논의된 결과를 요약해 보면 다음과 같다.

　　균형보험료는 담보위험의 보험계리적 現像인 순보험료와 보험계약자가 위험회피성향에 의해 추가적으로 부담하는 부가보험료와의 합으로 구성됨을 보였다. 여기서, 부가보험료는 명백히 투자소득의 함수로 나타나는데, 이 결과는 투자에 대한 기회비용이 수요와 공급에 의거한 합리적 보험요율에 영향을 주는 근본 요인임을 보여준다. 아울러, 보험수요자의 주식시장에의 참여가 최적보험수준에 미치는 영향을 Arrow-Pratt의 위험회피척도를 사용하여 분석하였다. 그러나 보다 일반적인 분석에서 투자위험이 담보위험과 동시적으로 존재하는 분석모형의 경우 Arrow-Pratt의 위험회피척도는 충분한 역할을 하지 못한다. 본고에서는 Ross에 의해 개발된 강력한 위험회피척도를 도입하여, 보험계약자의 최적보험과 최적포트폴리오에 미치는 위험회피深度의 효과 및 富의 효과를 도출하였다.

　　최근 보험산업환경은 금융산업 전반의 국제화, 자유화 및 개방화 조류를 타고 급변하고 있다. 소비자는 보험가격의 적정성에 대한 요구를 증대시키고 있으며 감독당국은 보험상품의 가격자유화를 시급한 현안 문제로 검토하고 있다. 본 논문의 연구결과는 경쟁에 의해 균형보험요율의 합리적 수준이 유지되기 위해서는 요율산정과정에 투자소득이 필수적으로 반영되어야 함을 지적하고 있다.

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Effect of Capital Market Return
On Insurance Coverage :
A Financial Economic Approach

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Recent financial theory views insurance policies as financial instruments that are traded in markets and whose prices reflect the forces of supply and demand. This article analyzes individual's insurance purchasing behavior along with capital market investment activities, which will provide a more realistic look at the tradeoff between insurance and investment in the individual's budget constraint. It is shown that the financial economic concept of insurance cost should reflect the opportunity cost of insurance premium. The author demonstrates the importance of riskless and risky financial assets in reaching an equilibrium insurance premium. In addition, the paper also investigates how the investment income could affect the four established theorems on traditional insurance literature. At the present time in Korea, the price deregulation is being debated as the most important current issue in insurance industry. In view of the results of this paper, insurance companies should recognize investment income in pricing their coverage if insurance prices are deregulated. Otherwise, price competition may force insurance companies to restrict coverage or to leave the market.

1. Introduction

There has been an emerging consensus on four fundamental aspects of optimal coverage of insurance in the traditional insurance economics:

(1)   If the insurance premium is actuarially fair to a risk-averse insured, he will buy full insurance, as shown in Arrow (1963, 1974), Smith (1968), Mossin (1968), and Ehrlich and Becker (1972).

(2)   If the insurance premium includes any positive loading factor which is proportionally related to the expected claim payment, it will never be optimal for the insured to take full insurance, as shown in Smith (1968) and Mossin (1968). That is, for the risk averse individual, it is always optimal to purchase only partial coverage of insurance with proportionate premium loadings. This implies that the individual will optimally share his risk with the insurance company.

(3)   Since the risk averse individual retains some risk under his optimal partial insurance contract, the optimal amount of insurance increases with the individual's degree of

risk aversion. That is, if the insured becomes more risk-averse in the Arrow-Pratt sense, he will purchase more coverage of insurance, as shown in Schlesinger (1981).

(4)   If the insured has decreasing absolute risk aversion in the Arrow-Pratt sense, insurance coverage will be an inferior good, which means that less is demanded as wealth increases, as shown in Mossin (1968) .

While the structure of these models cited above is very different, they all share with the assumption that the insurance contract is in isolation from other portfolio decisions in the individual's opportunity set. This assumption has often been called into question [Doherty (1981, 1984), Mayers and Smith (1983), Kahane and Kroll (1985), Smith and Buser (1987)]. Recent financial theory views insurance policies as financial instruments that are traded in markets and whose prices reflect the forces of supply and demand. If the individual treats insurance contracts as financial assets along with risky and riskfree assets in determining their optimal portfolio, the insurance purchase decision must be explicitly considered in the individual's budget constraint. The individual should determine whether the speculative risk of investing in the risky assets is acceptable, and if so, how his insurance decision covering insurable risk should be adjusted in response to a new source of risk, the speculative risk. In this respect, the economic concept of insurance cost should include alternative use of insurance premium or the opportunity cost of insurance premium. Any decrease in the insurance coverage can be used to finance an increase in the riskless bond or in the risky stock. This consideration may allow for different aspects of optimal insurance coverage from the early insurance literature. Both an insurable property-liability risk and other capital market risks are likely to have effects on the individual's insurance purchasing behavior.

The purpose of this paper is to construct a simplified financial theory of the demand for insurance under capital market uncertainty. In earlier related work, Witt and Hong (1992) developed insurance demand model which recognized the interdependent stochastic nature of the insured's insurance and investment activities. They assumed that a typical individual's investment opportunity set include all physical and real assets as well as financial assets, and as a consequence the positive/negative stochastic interdependence might exist between insurable risks and speculative risks. In essence, they provided a role of risky assets in risk management when the returns of marketable assets were stochastically correlated with insurable losses. In addition, it was shown that, unlike the results of the traditional insurance model, strict risk aversion is not sufficient to yield a positive insurance loading in the

presence of a speculative risk. That is, the sign and size of the insurance premium loading in addition to the actuarial value of the policy depend on the stochastic interdependent relationship between the insurable and speculative risks.

Unlike Witt and Hong (1992), the model developed here constrains the analysis to investment in financial markets. If the marketable assets are limited to financial assets in the capital market, empirical findings [e.g., Lambert and Hofflander (1966), Cummins and Nye (1980)] indicate that claims on property-liability insurance policies are not stochastically correlated with returns in financial market, so that assumption was adopted.[1] The analysis of this paper will provide a joint treatment of the insurance and capital market investment decisions. The literature on the demand for insurance and that on the demand for risky asset seem to have developed independently. Maybe the main reasons for avoiding to investigate their links have been due to the complexity of the problems and to the weakness of Arrow-Pratt measures of risk aversion in dealing with two sources of uncertainty simultaneously within a general expected utility framework. Recently, a series of recent papers by, among others, Ross (1981) and Kihlstrom et al (1981) propose a strong measure of risk aversion that is appropriate for use when there are multiple sources of uncertainties. Using the strong measure of risk aversion developed by Ross (1981), the model constructed in this paper will show that optimal insurance purchasing strategies critically depend on capital market return. The author demonstrates the importance of riskless and risky financial assets in reaching an equilibrium insurance premium. In essence, it is shown that the loadings in equilibrium insurance premium, which the individual is willing to pay in addition to actuarial value of the policy, reflect the capital market income which the individual recognize in investing financial assets. This result show that the financial-economic concept of insurance cost should consider the opportunity cost of insurance premium. In addition, the paper provides a formal analysis of how the simultaneous investment opportunities affect the four well-known theorems on insurance summarized above. The results of this paper supplements the traditional theories on insurance and suggest some extensions.

The paper is organized as follows. In Section 2, we introduce riskless and risky financial assets into the traditional insurable risk model. We derive the conditions for an optimal investment and insurance choice. Section 3 shows how the existence of capital market can affect optimal insurance decision. In Section 4, we will reexamine the Bernoulli

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[1]This kind of independent relationship between underwriting risk and investment risk has been frequently utilized in modelling the insurance company. See, among others, McCabe and Witt (1980), MacMinn and Witt (1987) and Kroll and Nye (1991).

criteria with and without premium loadings when insurance and investment decisions are simultaneous. In Section 4 and 5, we conduct two comparative statics analyses. The effect of increase in risk aversion as well as the effect of increase in wealth on simultaneous insurance and investment decisions are investigated respectively and the results are compared with those of existing literature. Section 6 summarizes our results and concludes the paper.

2. Assumption, Model and Its Solution

2.1. Notations

The following variables are defined.

W = the value of the endowed initial wealth;

R = one-plus-the-random-rate-of-return on a dollar invested in a portfolio of risky assets;

L = the value of the random amount of loss;

$f(R,L)$ = the joint probability density function for $0 \leq R < \infty$ and $0 < L < \infty$;

$f_1(R)$ = the marginal density of R;

$g_2(L|R)$ = the conditional density of L, that is, $f(R,L) = g_2(L|R)f_1(R)$;

$f_2(L)$ = the marginal density of L;

$g_1(R|L)$ = the conditional density of R, that is, $f(R,L) = g_1(R|L)f_2(L)$;

R_f = one-plus-the-interest-rate or one-plus-the-riskless-rate-of-return on a dollar invested in the safe asset;

q = the probability that a property-liability loss is incurred;

h = random variable denoting the risk associated with loss frequency. That is, $h=0$ with probability $1-q$ or $h=1$ with probability q;

a = the dollar amount invested in a portfolio of risky assets;

b = coefficient of coinsurance, $0 \leq b \leq 1$;

$p(b)$ = the premium for partial coinsurance coverage, b;

P = the premium for full insurance coverage, that is, $P = p(1)$;

Y = individual's final wealth at the end of the period;

Y_0 = individual's final wealth if a loss is not incurred, that is, $Y=Y_0$ with probability $1-q$;

Y_1 =   individual's final wealth if a loss is incurred, that is $Y=Y_1$ with probability q.

2.2. Model and Solutions

Based on the notations in the preceding sub-section, consider a risk averse individual with initial wealth, W. The individual has to allocate his wealth for investment or insurance purposes. First, the individual can buy insurance coverage b ($0 \leq b \leq 1$) for a premium p(b) where b is the coinsurance rate for insurance coverage. By assuming the coinsurance contracts, the insurance premium can be defined as the present value of expected indemnification amount, $\frac{1}{R_f}qE(L)$, plus the proportional loading, that is,

$$p(b) = b\,(1+\lambda)\,\frac{1}{R_f}\,qE(L), \text{ where } \lambda \geq 0.$$

If we denote the total premium of full insurance coverage by P, that is,

$$P = p(1) = (1+\lambda)\,\frac{1}{R_f}\,qE(L),$$

then the premium p(b) can be written as p(b) = bP. In the remainder of this paper, following this premium schedule, the insurance premium will be defined as (actuarially) "fair" if $\lambda=0$, or "unfair" if $\lambda>0$ (that is, the premium is said to be fair if $P = (1/R_f)qE(L)$, or unfair if $P > (1/R_f)qE(L)$). From now on throughout the paper, unless otherwise specified, loaded premium is assumed, that is

$$P > \frac{1}{R_f}\,qE(L). \tag{A1}$$

Next, suppose that the dollar amount of 'a' is invested in a portfolio of risky assets which will produce final stochastic return aR, and that the remainder of initial wealth W-a-bP can be used for investment in the riskless asset, which will produce final fixed return (W-a-bP)R_f. At the end of the period the individual's final wealth will be given by random amount Y_0 if no losses occur, where

$$Y_0 = (W-bP)R_f + a(R-R_f).$$

If losses do occur, the individual's random final wealth at the end of the period will be Y_1, where

$$Y_1 = (W-bP)R_f + a(R-R_f) - (1-b)L.$$

The random variable associated with insurable risk, L, and the random variable associated with speculative risk, R, are assumed to have a joint continuous probability distribution, $f(R,L)$. However, in the present model the marketable assets are limited to financial assets in the capital market. Corresponding to empirical support [Lambert and Hofflander (1966), Cummins and Nye (1980)], it is assumed that insurable risk and financial market risk are uncorrelated in that $E(L|R)$ is constant for all R.[2] That is,

$$E(L|R) = E(L) \text{ for all R,}$$

or

$$E(R|L) = E(R) \text{ for all L.} \tag{A2}$$

The assumption (A2) can be applied to a smaller class of pairs of insurable and speculative risks which are independently distributed each other. That is, the pair of random variables satisfying (A2) will belong to a wider class of pairs of random variables than those of independent random variables.

More simply, final wealth of this typical individual, Y, can be denoted by

$$Y = (W-a-bP)R_f + aR - h(1-b)L,$$

where the random variable h has the value of zero with the probability 1-q, or one with the probability q.

The individual chooses both investment and insurance level to maximize the expected utility of the final wealth. Let the individual's expected utility of the final wealth be $U(a,b)$ where:

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[2]Constant $E(L|R)$ for all R implies $Cov(R;L)=0$, but the converse is not true.

$$U(a,b) = E[u(Y)]$$

$$= (1-q) \int_{(L=0,\infty)} \int_{(R=0,\infty)} u(Y_0) \, f(R,L) \, dRdL$$

$$+ q \int_{(L=0,\infty)} \int_{(R=0,\infty)} u(Y_1) \, f(R,L) \, dRdL,$$

or equivalently,

$$U(a,b) = (1-q)E[u(Y_0)] + qE[u(Y_1)].$$

where $E[.]$ is the expectation operator.

We assume that $u'>0$ and $u''<0$, then U is concave due to the concavity of u. The first order conditions are given by:

$$D_1U(a,b) = (1-q)E[u'(Y_0)(R-R_f)] + qE[u'(Y_1)(R-R_f)] = 0, \tag{1}$$

$$D_2U(a,b) = -(1-q)PR_fE[u'(Y_0)] + qE[u'(Y_1)(-PR_f+L)] = 0, \tag{2}$$

where $D_iU(a,b)$ denote the partial derivative of U with respect to ith argument.[3]

First, we will consider the optimal condition with respect to investment decision. Before proceeding discussions, one important observation should be made here. The individual will buy a positive amount of risky assets regardless of insurance decision, if $D_1U(0,b) > 0$. When $a = 0$, one obtains $Y_0 = W-PR_f$ and $Y_1 = W-PR_f-(1-b)L$. Now, the optimal condition (1) may be reduced to

$$D_1U(0,b) = E(R-R_f) \{(1-q)E[u'(Y_0)] + qE[u'(Y_1)]\} + q \, Cov[u'(Y_1);R].$$

One obtains $D_1U(0,b) > 0$ if and only if $E(R)>R_f$, since $Cov[u'(Y_1);R] = 0$ when a=0. In this case, the net positive return on risky asset, $E(R)>R_f$, will be a necessary and sufficient condition for purchasing certain amount of risky asset. It may be noted that the condition

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[3]For the rigorous second-order condition for an interior maximum, see Witt and Hong (1992). Throughout the paper, it will be assumed that these second order conditions are satisfied so that some optimal values, a^* and b^* exist where $-\infty < a^* < \infty$, and $-\infty < b^* < \infty$.

$E(R) > R_f$ is equivalent to the boundary condition for positive investment in risky asset derived in Arrow (1963). From now on throughout this paper, we will maintain the assumption that the net expected return on risky investment is positive, that is,

$$E(R) > R_f. \tag{A3}$$

The first order condition (1) with respect to 'a' can be rearranged as follows:

$$E(R) = R_f - \frac{(1-q)\text{Cov}[u'(Y_0);R] + q\,\text{Cov}[u'(Y_1);R]}{(1-q)E[u'(Y_0)] + qE[u'(Y_1)]}. \tag{3}$$

where Cov[.] denotes the covariance operator. The negative of the second term on the RHS of (3), which is the marginal risk premium for risky investment, characterizes the effect of risk aversion on the individual's risky asset choice.[4] It should be noted that, unlike the result of Witt and Hong (1992), marginal risk premium is always positive given the assumption $E(L|R) = E(L)$ for all R since $u'(Y_0)$ and $u'(Y_1)$ is negatively related with R under risk aversion ($u''<0$). To see this, note that $\text{Cov}[u'(Y_1);R] = E\{u'(Y_1)\,[R-E(R)]\}$. Let the function

$$V = E[u'(Y_1)]|_{R=E(R)}.$$

Since u' is decreasing in Y_1 and Y_1 is increasing in R, one obtains the following inequalities $E[u'(Y_1)|R] < V$, if $R > E(R)$. It follows immediately that

$$E[u'(Y_1)|R]\,[R-E(R)] < V\,[R-E(R)] \tag{4}$$

if $R > E(R)$. In reality, the above inequality in (4) holds for all R. For $0 \leq R \leq E(R)$, one obtains $E[u'(Y_1)|R] > V$, but multiplication by R-E(R) will still make the inequality in (4) hold. Taking expectations with respect to R on both side of (6) yields

$$E\{u'(Y_1)]\,[R-E(R)]\} < V\,E[R-E(R)] = 0.$$

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[4]To see why the negative of the second term on the RHS of (3) is the marginal risk premium, refer to Witt and Hong (1992). Note that for a risk neutral individual, $u'(Y_0)$ or $u'(Y_1)$ would be a constant so that the second term on the RHS of (3) would be reduced to zero.

Thus, $Cov[u'(Y_1);R]$ is negative and so marginal risk premium is positive. Thus, the individual will invest in the risky asset up to the point where expected marginal rate of return for the risky asset equals the riskfree rate plus the marginal risk premium.

The first order condition (2) for optimal insurance can be rewritten as

$$P = \frac{1}{R_f} \{qE(L) + \frac{q(1-q)E(L)\{E[u'(Y_1)]-E[u'(Y_0)]\} + qCov[u'(Y_1);L]}{(1-q) E[u'(Y_0)] + q E[u'(Y_1)]}\}. \tag{5}$$

The first term in (5), $(1/R_f)qE(L)$, the actuarial value of the policy. The second term in (5) characterizes the effect of, among others, capital market return on the equilibrium premium concept. It should be noted that the second term is always positive, since $u'(Y_0) <$ $u'(Y_1)$ given risk aversion ($u''<0$) and $Cov[u'(Y_1);L] > 0$ given the assumption (A2).[5] The positive second term in (5) allows the risk averse individual for a positive loading in addition to the present value of the expected loss indemnification. Therefore, the risk averse individual will choose the level of insurance coverage such that his marginal premium equals the present value of the expected marginal indemnification plus the absolute value of marginal risk premium for his insurance contract. The second term in (5) shows how insurance and investment opportunities are related to prevailing insurance contract prices. The size of premium loading depends critically on the investment return. In general, it will be impossible to separate both decisions without further critical restrictions even when insurable and speculative risks are stochastically independent.

3. The Effect of Financial Assets on the Demand for Insurance

A question of some importance is whether the existence of capital markets will induce the risk averse individual to buy more or less insurance than it would without those markets. It is not immediately apparent whether the optimal level of insurance will increase or decrease according to new speculative risk in the portfolio without additional information. The whole risk and return of final wealth should further be adjusted to maximize individual's expected utility. The following proposition formally shows how the addition of risky asset in the insurable risk portfolio affects the optimal insurance coverage. As we shall see, the direction of change of optimal coverage depends on the characterization of risk aversion. From now

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[5]$Cov[u'(Y_1);L] > 0$ can be easily derived by following the similar procedure showing $Cov[u'(Y_1);R] < 0$, described earlier.

on, the statement "decreasing, constant or increasing absolute risk aversion in the Arrow-Pratt sense" is condensed as $DARA_{AP}$, CARA or $IARA_{AP}$, respectively.[6]

Proposition 1.

If a risky asset is introduced in the insurable risk portfolio, then the effect on the optimal amount of insurance is

(1)   negative if u exhibits $DARA_{AP}$;

(2)   null if u exhibits CARA;

(3)   positive if u exhibits $IARA_{AP}$.

Proof. Define the function K(a,b) as follows:

$$K(a,b) = (1-q) \int\int u'(Y_0)(-PR_f)\ f(R,L)\ dRdL + q \int\int u'(Y_1)(-PR_f+L)\ f(R,L)\ dRdL, \quad \cdots (6)$$

where the RHS of (6) is $D_2U(a,b)$ in equation (2). Since D_2K is negative, it follows that by the Implicit Function Theorem there exists a function k such that

$$K(a,k(a)) = 0 \text{ and } k'(a) = -\frac{D_1K}{D_2K},$$

where

$$D_1K(a,b) = (1-q) \int\int u''(Y_0)(-PR_f)(R-R_f)\ f(R,L)\ dRdL$$

$$+ q \int\int u''(Y_1)(-PR_f+L)(R-R_f)\ f(R,L)\ dRdL.$$

Let b^o denote the optimal value of b when a=0. Then given loaded premium in assumption (A1), the range of b^o will be restricted to $0 \le b^o < 1$.[7] The sign of $k'(0) = -D_1K(0,b^o)/D_2K(0,b^o)$ will be examined. Since $D_2K(0,b^o)<0$, the sign of $k'(0)$ is equal to that of $D_1K(0,b^o)$. If $(a,b) = (0,b^o)$, then $Y_0 = (W-b^oP)R_f$ and $Y_1 = (W-b^oP)R_f - (1-b^o)L$. Now, $D_1K(0,b^o)$ can be rewritten as

$$D_1K(0,b^o) = (1-q)\ u''(Y_0)(-PR_f) \int\int (R-R_f)\ f(R,L)\ dRdL$$

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[6]It also may be noted that Arrow-Pratt characterization of constant absolute risk aversion (CARA) is identical to Ross characterization, which will be frequently appear later, corresponding to the exponential utility function such that u(W) = - exp(-cW). Thus, the subscript $_{AP}$ will be omitted for CARA case.

[7]This will be clearly shown in Proposition 2 in the next section.

$$+ q \int (R-R_f) \, g_1(R|L)dR \int u''(Y_1)(-PR_f+L) \, f_2(L)dL.$$

Since we assumed $E(R|L)=E(R)$ for all L, $D_1K(0,b°)$ can be reduced to

$$D_1K(0,b°) = [E(R)-R_f]\{(1-q)u''(Y_0)(-PR_f) + qE[u''(Y_1)(-PR_f+L)]\}.$$

Define the function B as

$$B = - \frac{u''(Y_1)}{u'(Y_1)} \Big|_{L=PR_f}$$

Given DARA$_{AP}$, one obtains

$$-\frac{u''(Y_1)}{u'(Y_1)} > B \tag{7}$$

if $L \geq PR_f$. From (7), it follows immediately that

$$u''(Y_1)(L-PR_f) < - B \, u'(Y_1)(L-PR_f), \tag{8}$$

if $L \geq PR_f$. In reality, the inequality of (8) holds also for $0 \leq L \leq PR_f$. Taking expectations on both side of (8), and noting that B is a fixed number, one obtains

$$E[u''(Y_1)(L-PR_f)] \leq -B \, E[u'(Y_1)(L-PR_f)]. \tag{9}$$

Again, since $Y_1<Y_0$, given DARA$_{AP}$ one obtains the following inequality

$$- u''(Y_0) < B \, u'(Y_0). \tag{10}$$

Combining (9) and (10) gives

$$K_1(0,b°) = [E(R)-R_f]\{(1-q)u''(Y_0)(-PR_f) + qE[u''(Y_1)(-PR_f+L)]\}$$

$$< -B \, [E(R)-R_f]\{-PR_f(1-q)E[u'(Y_0)] + qE[u'(Y_1)(L-PR_f)]\}$$

$$= 0.$$

Therefore, the result is $k'(0) < 0$ under $DARA_{AP}$. Employing similar procedure we can obtain $k'(0) = 0$ under CARA, and $k'(0) > 0$ under $IARA_{AP}$.   Q.E.D.

Thus, if a risky asset, whose net expected return is positive, is introduced to the opportunity set of the individual who initially holds optimal insurance coverage in the absence of risky assets, the individual will always buy certain positive amount of risky asset[8], and simultaneously he adjusts insurance coverage according to his risk preference. Arrow (1965), among others, has argued that individuals are characterized by decreasing absolute risk aversion. If this is a reasonable assumption, the existence of capital markets induces the individual to demand less insurance than it would have in the absence of such a market.

4. Bernoulli Principle

In this section, the Bernoulli Principle will be considered. In the classic theory of demand for insurance, it would be optimal for the insured to take full insurance if the insurance premium includes no loading [Arrow (1964, 1974), Smith (1968), Mossin (1968)]. Recently, Doherty and Schlesinger (1983b) examined the choice of deductible insurance in the presence of uninsurable background risk and derived more general results that the Bernoulli principle holds if random initial wealth is independently distributed from the insurable loss. The following proposition is an extension of the results obtained by Doherty and Schlesinger (1983b). The proposition shows that even if the insurable and speculative risks have joint distribution rather than independent distribution, but there exist no interdependence in (A2), the Bernoulli principle still holds.

Proposition 2.
(1)   Full coverage (b=1) is optimal if and only if $P=(1/R_f)qE(L)$;
(2)   Partial coverage $(0 \leq b° < 1)$ is optimal if and only if $P>(1/R_f)qE(L)$.

Proof. When $b=1$, $Y_0 = Y_1 = Y = (W-PR_f)+a(R-R_f)$. For arbitrary a, one obtains

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[8]Recall that the individual invests in risky asset if and only if $E(R)>R_f$.

$D_2U(a,1)$ = $(1-q) \int\int u'(Y)(-PR_f) f(R,L) dRdL + q \int\int u'(Y)(-PR_f+L) f(R,L) dRdL$

= $\int u'(Y)(-PR_f) (\int f(R,L)dL) dR + q \int u'(Y) f_1(R) \int L g_2(L|R)dL dR$

= $E[u'(Y)][-PR_f+qE(L)]$.

Thus, for any investment portfolio $D_2U(a,1)=0$ if and only if $P=(1/R_f)qE(L)$, or $D_2U(a,1)<0$ if and only if $P>(1/R_f)qE(L)$.   Q.E.D.

An intuitive interpretation of Proposition 2 can be given in terms of the whole risk of final wealth. Notice that if the costs of insurance are actuarially fair $(P=(1/R_f)qE(L))$, the mean final wealth,

$$E(Y) = WR_f + a[E(R)-R_f] - qE(L),$$

remains constant regardless of all insurance decisions. However, higher moments of final wealth distribution will be changed. More generally the proposition can be explained by the riskiness of final wealth distribution developed by Rothschild and Stiglitz (1970).

Corollary 1.
Assume that $P=(1/R_f)qE(L)$. Then the final wealth with partial insurance is riskier in the Rothschild and Stiglitz sense than that with full insurance.

Proof. Let $Y(a,b)$ be random variable denoting individual's final wealth, that is, $Y(a,b)$ = $WR_f + a(R-R_f) - bPR_f - h (1-b)L$. Now, we will show that $Y(a,b^*)$ where $0 \le b^* < 1$ is a mean preserving spread of $Y(a,1)$, which is equivalent to saying that there exist a random variable Z satisfying $E(Z|Y(a,1)) = 0$ for all $Y(a,1)$ such that $Y(a,b^*) \overset{d}{=} Y(a,1)+Z$.[9] To see this, $Y(a,b^*)$ can be written as

$$Y(a,b^*) = [W-b^*P]R_f + a(R-R_f) - h (1-b^*)L$$

$$= [W-P]R_f + a(R-R_f) + (1-b^*) (PR_f - hL)$$

$$= Y(a,1) + (1-b^*)Z,$$

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[9] $\overset{d}{=}$ means "has the same distribution as". See Rothschild and Stiglitz (1970).

where $Z \equiv (PR_f-hL)$, and $E\{Z|Y(a,1)\}=0$ since $E(hL|R) = qE(L)$. Thus, any decrease in b from one to b^* where $0 \leq b^* < 1$ represents a mean preserving spread.   Q.E.D.

Since, given fair premium, final wealth distribution with partial insurance has the same distribution as that with full insurance plus some noise, every risk averse individual will prefer full coverage.

Now, we will consider the part (2) of Proposition 2. In the classic theory of insurance, it is well-known that less than full coverage is optimal if the insurance premium includes any proportional loading [Arrow (1964, 1974), Smith (1968), Mossin (1968)]. In the present model where investment and insurance decisions are simultaneous these results continue to hold. Partial insurance with positive loading is consistent with the Bernoulli criterion. A reason why full insurance cannot be an optimal choice given loaded premium can be explained by concept of mean preserving spread. If premium is loaded, then the existence of capital market can always serve to reduce the final wealth fluctuation of full insurance by moving to partial coverage of insurance while expected wealth is unaltered, as shown in Witt and Hong (1992). That is, the individual holding full insurance can always improve his expected utility by reducing to partial coverage if capital asset is available.

5. The Effect of an Increase in Risk Aversion

In this section, the effect of an increase in risk aversion on optimal insurance coverage will be considered. Traditionally, the optimal amount of insurance has been regarded to be a directly increasing function of the individual's degree of the risk aversion in the Arrow-Pratt sense [see e.g., Schlesinger (1981)]. One might reasonably expect that if u_A is more risk averse than u_B then, b_A, the optimal coverage of insurance for u_A, is larger than b_B, the optimal coverage of insurance for u_B. However, this result does not follow from the Arrow-Pratt risk aversion ordering in the present model since an individual's wealth was the sum of two random variables.[10] However, if the Arrow-Pratt measures of risk aversion is replaced by strong measures of risk aversion of Ross (1981), then unambiguous result can

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[10]Ross(1981) and Kihlstrom et al (1981) provided more concrete examples in which the Arrow-Pratt measure of risk aversion would violate our intuition when an individual's final wealth is the sum of two independent random variables. The intuitive result (that is, $b_A \geq b_B$) does not follow in the present model from the Arrow-Pratt ordering, as shown in Hong (1992).

be obtained. For the sake of completeness we will begin by recalling the definition and basic theorems of of Ross (1981). Consider two von-Neumann Morgenstern utility functions u_A and u_B.

Ross Definition 1.

The statement "u_A is more risk averse than u_B in Ross sense" is condensed to $u_A \geq_R u_B$ and is defined as follows: $u_A \geq_R u_B$ if and only if there exist a $\mu(>0)$ such that

$$\frac{u_A{}'(W_1)}{u_B{}'(W_1)} \leq \mu \leq \frac{u_A{}''(W_2)}{u_B{}''(W_2)} ,$$

for all W_1 and W_2, where W_1 and W_2 are different wealth levels.

To put these definitions to use, we will adopt the following theorem of Ross (1981).

Ross Theorem 1.

The following three conditions are equivalent:

(1) $u_A \geq_R u_B$.
(2) There exist a function H such that $u_A = \mu u_B + H$, where $\mu > 0$, $H' \leq 0$ and $H'' \leq 0$.

One important relationships between the Ross and the Arrow-Pratt definition should be noted. The Ross definition is indeed strictly stronger than the Arrow-Pratt ordering in the sense that $u_A \geq_R u_B$ implies $u_A \geq_{AP} u_B$, but the converse is not true.[11]

In the following Lemma 1 and Lemma 2, two non-simultaneous results will be derived and interpreted. Utilizing these lemmas, the simultaneous effects of an increase in risk aversion on insurance and investment decision are considered in Proposition 3.

The following type of individual's nonsimultaneous behavior in Lemma 1 can be frequently observed in the real world. For example, this may be the corresponding situation where some of initial wealth of the individual is held in riskless or risky financial assets. In this case initial wealth is uncertain. Recently Doherty and Schlesinger (1983b) and Turnbull (1983) introduced random initial wealth into the traditional insurable risk model, and analyzed the impact of increase in risk aversion on insurance coverage. Hence, the

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[11]See theorem 2 of Ross (1981).

nonsimultaneous result in Lemma 1 is reminiscent of Doherty and Schlesinger's (1983) or Turnbull's (1983) results. However, the Lemma 1 is a generalization of Schlesinger (1983) or Turnbull (1983) in that the pair of random variables in (A2) will belong to a wider class of pairs of random variables than that of independent random variables, which are employed in Doherty and Schlesinger (1983) or Turnbull (1983).

Lemma 1.

For the given investment portfolio, an increase in risk aversion in Ross sense will increase the optimal coverage of insurance. That is,

$$u_A \geq_R u_B \Rightarrow b_A \geq b_B.$$

Proof: The first-order condition with respect to b for u_B for the given investment portfolio can be written as

$$D_2 U_B(a, b_B) = -PR_f\{(1-q)E[u_B'(Y_0^B)] + qE[u_B'(Y_1^B)]\} + qE[u_B'(Y_1)L] = 0, \qquad (11)$$

where

$$Y_0^B = (W - b_B P)R_f + a(R - R_f),$$

$$Y_1^B = (W - b_B P)R_f + a(R - R_f) - (1 - b_B)L.$$

It will be shown that

$$D_2 U_A(a, b_B) \geq 0,$$

which implies that $b_A \geq b_B$. Using (11) and Ross Theorem 1, one obtains

$$D_2 U_A(a, b_B) = -PR_f\{(1-q)E[H'(Y_0^B)] + qE[H'(Y_1^B)]\} + qE[H'(Y_1^B)L],$$

or equivalently,

$$D_2 U_A(a, b_B) = -PR_f\{(1-q)E[H'(Y_0^B)] + qE[H'(Y_1^B)]\} + qE[H'(Y_1^B)]E(L)$$

$$+ qCov[H'(Y_1^B); L]. \qquad (12)$$

First, it will be shown that the summation of the first and second term of the RHS of (12) is positive. Since $Y_0^B > Y_1^B$ and $H'' \leq 0$, one obtains $H'(Y_0^B) \leq H'(Y_1^B)$ and $-H'(Y_0^B) \geq -H'(Y_1^B)$. Using these relationships, one obtains

$$-PR_f(1-q)E[H'(Y_0^B)] \geq -PR_f(1-q)E[H'(Y_1^B)]. \tag{13}$$

Using (13), the summation of the first and second terms in (12) can be signed as follows.

$$- PR_f \{(1-q)E[H'(Y_0^B)] + qE[H'(Y_1^B)]\} + qE[H'(Y_1^B)]E(L)$$

$$\geq - PR_f \{(1-q)E[H'(Y_1^B)] + qE[H'(Y_1^B)]\} + qE[H'(Y_1^B)]E(L)$$

$$= E[H'(Y_1^B)][-PR_f+qE(L)]$$

$$\geq 0.$$

Now, it will be shown that the third term in (12), $Cov[H'(Y_1^B);L]$, is positive. Note that

$$Cov[H'(Y_1^B);L] = E\{H'(Y_1^B) [L-E(L)]\}.$$

Define the function V as

$$V = E[H'(Y_1^B)]|_{L=E(L)}.$$

Since H' is decreasing in Y_1^B and Y_1^B is decreasing in L, one obtains $E[H'(Y_1^B)|L]>V$, if $L \geq E(L)$. It follows immediately that

$$E[H'(Y_1^B)|L] [L-E(L)] > V [L-E(L)] \tag{14}$$

if $L \geq E(L)$. In reality, the above inequality in (14) holds for $0 \leq L \leq E(L)$. Taking expectations with respect to L on both side of (14) yields

$$E\{H'(Y_1^B)] [L-E(L)]\} > V E[L-E(L)] = 0.$$

Thus, $\text{Cov}[\text{H}'(\text{Y}_1{}^\text{B});\text{L}]$ is positive. This completes the proof. Q.E.D.

For the given risky investment portfolio, the Ross measure of risk aversion is sufficient to describe the individual's behavior in choosing the optimal insurance coverage without further restricting utility functions. If the individual becomes more risk averse in Ross sense, more coverage of insurance will be purchased with reducing the expected final wealth as well as the risk of final wealth distribution.

Now, in the following Lemma 2, the impact of an increase in risk aversion on the optimal proportion of risky assets in the optimal investment portfolio will be considered when the rate of insurance coverage is given. The individual may encounter these kinds of situations where the investment decision has to be made in the presence of the insurance decision. This case may be approximated to the case where some individual risks should be covered by compulsory insurance laws. For example, in most states of U.S. the owners and operators of automobiles have to carry automobile liability insurance at least equal to a certain amount by the compulsory insurance laws before the automobile can be registered and licensed.[12] The other example will be the flood insurance. Federal law requires flood insurance for any real property located in a flood zone.

Lemma 2.
For a given insurance coverage, an increase in risk aversion in Ross sense reduce the optimal level of risky investment; that is,

$$u_A \geq_R u_B \Rightarrow a_A \leq a_B.$$

Proof: Since the method of proof is similar to that of Lemma 1, the proof is omitted here.

When R and L are stochastically non-interdependent, the risk of final wealth is increasing with the amount of risky investments if other things being equal. Thus, the more risk averse individual, who retains the property-liability risk, optimally reduces the level of risky investments below the level of that for the less risk averse individual. That is, the more

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[12]At the present time more than half of the states have enacted some type of compulsory automobile liability insurance law as a condition for driving within the state. Other example will involve the some compulsory insurance for professional liability. According to Faure and Van den Bergh (1989), Belgium law embraces compulsory insurance for more than 40 activities. Among these, compulsory insurance is required for professional liability of architects and attorneys. In recent years compulsory insurance is also debated as a remedy to liability problems in the medical profession. Faure and Van den Bergh (1989) provide the discussion whether liability insurance for professional services should be made compulsory or not.

risk averse individual will choose a less risky investment portfolio with a lower expected
final wealth, other things being equal.

Now, it will be natural to ask whether or not these two non-simultaneous results will
still be effective when the individual determines his coverage of insurance and the
composition and size of his investment portfolio simultaneously. Using the results of Lemma
1 and Lemma 2, the simultaneous impact of an increase in risk aversion are provided in the
following Proposition 3.

Proposition 3.

An increase in absolute risk aversion in Ross sense increases the amount of insurance but
reduces the amount of risky assets if $D_{12}U \leq 0$; that is,

$$\left. \begin{array}{l} u_A \geq_R u_B \\ D_{12}U \leq 0 \end{array} \right\} \quad \Rightarrow \quad \left\{ \begin{array}{l} a_A \leq a_B \\ b_A \geq b_B. \end{array} \right.$$

Proof: Denote the impact of a marginal increase in a absolute risk aversion in Ross sense on
the optimal values of a and b by $\partial a/\partial ARA_R$ and $\partial b/\partial ARA_R$. By totally differentiating the
first order conditions (1) and (2) with respect to ARA_R and solving the derivatives for
$\partial a/\partial ARA_R$ and $\partial b/\partial ARA_R$ by Cramer's Rule, one obtains

$$\frac{\partial a}{\partial ARA_R} = \frac{D_{12}U \ D_{2ARA_R}U - D_{1ARA_R}U \ D_{22}U}{|H|},$$

$$\frac{\partial b}{\partial ARA_R} = \frac{D_{21}U \ D_{1ARA_R}U - D_{2ARA_R}U \ D_{11}U}{|H|},$$

where and $D_{12}U = \partial^2 E[u(Y)]/\partial a \partial b$, $D_{1ARA_R}U = \partial^2 E[u(Y)]/\partial a \partial ARA_R$, and so on. Now,
we need to evaluate the sign of $\partial a/\partial ARA_R$ and $\partial b/\partial ARA_R$ for an increase in ARA_R, starting
from initial optimal points of a and b, that is, $a = a_B$, $b = b_B$. Our non-simultaneous results
imply that $D_{1ARA_R}U(a_B,b_B) \leq 0$ by Lemma 1 and $D_{2ARA_R}U(a_B,b_B) \geq 0$ by Lemma 2.
Since $D_{11}U(a_B,b_B)$ and $D_{22}U(a_B,b_B)$ are negative, $\partial a/\partial ARA_R$ and $\partial b/\partial ARA_R$ have the
same signs as $D_{1ARA_R}U(a_B,b_B)$ and $D_{2ARA_R}U(a_B,b_B)$ respectively if $D_{12}U(a_B,b_B) \leq$
0.

Q.E.D.

This proposition suggests that although the positive effects of an increase in risk aversion on insurance coverage is possible, they do not generally hold when the individual makes simultaneous insurance and investment decisions. To derive the unanimous simultaneous result, it must be assumed that risky investment and insurance work like stochastic substitutes for each other ($D_{12}U<0$) or insurance and investment decisions are totally independent ($D_{12}U=0$). If $D_{12}U \leq 0$, an increase in risk aversion in Ross sense increases the coverage of insurance (b), and simultaneously reduces the size of risky assets (a) in the optimal investment portfolio. The assumption, $D_{12}U \leq 0$, is crucial in deriving unambiguous simultaneous results because the results of Proposition 3 do not necessarily hold when $D_{12}U > 0$. This can be contrasted with the previous results in the insurance literature.

Traditionally, optimal amount of insurance has been regarded to be a directly increasing function of the individual's degree of the risk aversion in the Arrow-Pratt sense [see e.g., Schlesinger (1981)]. Recently, Doherty and Schlesinger (1983b) and Turnbull (1983) introduced random initial wealth into the traditional insurable risk model, and analyzed the impact of increase in risk aversion on insurance coverage. The results of Doherty and Schlesinger (1983) and Turnbull (1983) have been driven by the implicit assumption that the insurance decision would be made after all the other investment decisions are completed. Thus, it is not surprising that their results are similar to those of traditional results where the simultaneous investment opportunity has been ignored, if the Arrow-Pratt measure of risk aversion is replaced by the stronger measure of risk aversion of Ross (1981) or Kihlstrom et al (1981). The results of our model include the results of Schlesinger (1983) and Turnbull (1983) as special cases. First, when the insurance decision can be made after the investment decisions, insurance can be regarded as an increasing function of the individual's degree of risk aversion. Second, when decisions are simultaneous, insurance can be regarded as an increasing function of the individual's degree of the risk aversion if risky investment and insurance are independent ($D_{12}U=0$) or work like stochastic substitutes for each other ($D_{12}U<0$).

6. Wealth Effect

Decreasing absolute risk aversion (DARA) has been regarded as a normative hypothesis for an individual's economic behavior. In a well-known article, Mossin (1968)

showed that insurance was an inferior good, which was demanded less with larger wealth, under DARA in the Arrow-Pratt sense. However, unfortunately no attempts were made to investigate the wealth effects within two sources of uncertainty model, for example, employed in Turnbull (1983) or Doherty and Schlesinger (1983). We may fill this gap. The following results of Lemma and Proposition are new to the existing insurance literature.

Generally in a situation in which individual's final wealth is the sum of two random variables, the wealth effect on optimal amount of insurance is ambiguous with the Arrow-Pratt characterization of increasing or decreasing risk aversion. We will show the meaningful results utilizing Ross (1981) notions of the increasing/decreasing absolute risk aversion.

The following strong concept of increasing/decreasing absolute risk aversion are defined by Ross(1981).

Ross Definition 2.
The statement "decreasing (increasing) absolute risk aversion in Ross sense" is condensed as $DARA_R$ ($IARA_R$), and the utility function, u, displaying decreasing (increasing) absolute risk aversion in Ross sense is defined as follows:

$$u \text{ exhibits } DARA_R \text{ iff } u(x) \geq_R u(x+y) \text{ for any x and y} > 0,$$

$$u \text{ exhibits } IARA_R \text{ iff } u(x+y) \geq_R u(x) \text{ for any x and y} > 0.$$

These definitions can be easily put to use by applying the following theorem.

Ross Theorem 2.
(1)   u exhibits $DARA_R$ iff there exist c such that $\dfrac{u'''(x)}{u''(x)} \leq c \leq \dfrac{u''(x)}{u'(x)}$ for any x;

(2)   u exhibits $IARA_R$ iff there exist c such that $\dfrac{u'''(x)}{u''(x)} \geq c \geq \dfrac{u''(x)}{u'(x)}$ for any x.

The conditions of above Ross Theorem can be expressed in terms of the Arrow-Pratt coefficients of absolute risk aversion. For the case of $DARA_R$, let

$$ARA_{AP} = - \frac{u''(x)}{u'(x)},$$

then

$$\frac{\partial}{\partial x} ARA_{AP} = \frac{\partial}{\partial x} [-\frac{u''(x)}{u'(x)}] = -\frac{u'(x)u'''(x) - [u''(x)]^2}{[u'(x)]^2} = -\frac{u'''(x)}{u'(x)} + \frac{[u''(x)]^2}{[u'(x)]^2},$$

or equivalently,

$$\frac{\partial}{\partial x} ARA_{AP} = ARA_{AP} [\frac{u'''(x)}{u''(x)} - \frac{u''(x)}{u'(x)}]. \tag{15}$$

Therefore, if there exist a real number c such that the conditions of $DARA_R$ ($IARA_R$) in Ross Theorem 2 hold, ARA_{AP} is decreasing (increasing).

Now, utilizing Ross definitions and theorems presented above, we can analyze the following unambiguous wealth effect on insurance coverage.

Lemma 3.
For a given risky investment portfolio, the effect on the amount of insurance coverage from an increase in initial wealth is

(1)   negative if u exhibits $DARA_R$; that is,

(2)   null if u exhibits CARA; that is,

(3)   positive if u exhibits $IARA_R$.

Proof: Let the function $K(a,W)$ be defined by the RHS of (2). Since D_1K is negative by the second order condition, there exists a function k such that

$$K(k(W), W) = 0 \text{ and } k'(W) = -\frac{D_2K}{D_1K},$$

where

$$D_2K(k(W),W) = -(1-q)P(R_f)^2 E[u''(Y_0)] + qR_f E[u''(Y_1)(-PR_f+L)].$$

Note that $k'(W)$ is greater or less than zero as D_2K is greater or less than zero. First, it will be shown that $D_2K(k(W),W)<0$ under $DARA_R$, which is part (1) of this proposition. In consideration of part (1) of Ross Theorem 2 and equation (15), it follows that

$$\frac{\partial}{\partial Y_1}\{-\frac{E[u''(Y_1)|L]}{E[u'(Y_1)|L]}\} < 0, \tag{16}$$

if u exhibits $DARA_R$. Define the function B as

$$B = -\frac{E[u''(Y_1)|L]}{E[u'(Y_1)|L]}\Big|_{L=PR_f} \tag{17}$$

Since - $E[u''(Y_1)|L]/E[u'(Y_1)|L)]$ are decreasing in Y_1 under $DARA_R$, the following inequalities can be obtained from (16) and (17)

$$-\frac{E[u''(Y_1)|L]}{E[u'(Y_1)|L]} \geq B,$$

or equivalently

$$E[u''(Y_1)|L](L-PR_f) \leq -BE[u'(Y_1)|L](L-PR_f), \tag{18}$$

if $L \geq PR_f$. In reality, the inequality of (18) also holds for $0 \leq L \leq PR_f$. Taking expectations with respect to L on both sides of (18) and noting that B is a fixed number, one obtains

$$E[u''(Y_1)(L-PR_f)] \leq -B\,E[u'(Y_1)(L-PR_f)]. \tag{19}$$

Again, since $Y_0 > Y_1$, one obtains the following inequality

$$-E[u''(Y_0)] \leq B\,E[u'(Y_0)], \tag{20}$$

under $DARA_R$. By combining (19) and (20), one obtains

$$D_2K(k(W),W) = -(1-q)P(R_f)^2E[u''(Y_0)]+qR_fE[u''(Y_1)(-PR_f+L)]$$

$$\leq \quad -B\,R_f\{-PR_f(1-q)E[u'(Y_0)] + qE[u'(Y_1)(L-PR_f)]\}$$

$$= \quad 0.$$

Therefore, the result for the case of $DARA_R$ was proved. The CARA result directly follows from the definition of CARA, $c = -u''(Y)/u'(Y)$, where c is some constant. For the proof of $IARA_R$ result, it requires similar procedure employed in the proof of the $DARA_R$ result.

<div align="right">Q.E.D.</div>

An increase in the value of initial wealth decreases the level of insurance coverage under $DARA_R$, has no effect on the level of insurance coverage under CARA, or increases the level of insurance coverage under $IARA_R$. Thus, for a given risky investment portfolio, the individual adjust the amount of insurance through changes in the level of the retained amount of insurable risk as the value of initial wealth changes. A more intuitive interpretation of Lemma 3 can be given. For example, for a case of $DARA_R$, an increase in the value of initial wealth decreases the degree of risk aversion as defined in Ross Theorem 2, and so the individual can improve expected utility by reducing the level of insurance coverage, b, as shown in Corollary 1. The interpretation of $IARA_R$ and CARA results follows a similar argument.

Now, the impact of an increase in initial wealth value on the size of risky investment portfolio will be examined when the coverage of insurance is fixed.

Lemma 4.

For a given coverage of insurance, the effect on the proportion of risky assets in the optimal investment portfolio from an increase in the value of initial wealth is

(1)   positive if u exhibits $DARA_R$; that is,

(2)   null if u exhibits CARA; that is,

(3)   negative if u exhibits $IARA_R$; that is,

Proof: Let W_0 and W_1 denote two different initial wealth where $W_1 > W_0$ and let a_0 and a_1 denote optimal level of a under W_0 and W_1 for the given coverage of insurance, b, respectively, that is,

$$D_1U(a_0,b)|_{W=W_0} = 0, \text{ or } D_1U(a_1,b)|_{W=W_1} = 0.$$

Now, it will be shown that $D_1U(a_1,b)|_{W=W_0} < 0$ under $DARA_R$, which implies that $a_0 < a_1$, or $\partial a/\partial W > 0$. To see this, define the following notations

$$Y_0{}^0 = (W_0\text{-}bP)R_f + a_1(R\text{-}R_f),$$

$$Y_1{}^0 = (W_0\text{-}bP)R_f + a_1(R\text{-}R_f) - (1\text{-}b)L,$$

$$Y_0{}^1 = (W_1\text{-}bP)R_f + a_1(R\text{-}R_f),$$

$$Y_1{}^1 = (W_1\text{-}bP)R_f + a_1(R\text{-}R_f) - (1\text{-}b)L.$$

Using these notations, one obtains

$$u(Y_0{}^0) \geq_R u(Y_0{}^1), \text{ and } u(Y_1{}^0) \geq_R u(Y_1{}^1),$$

or equivalently,

$$u(Y_0{}^0) = \mu u(Y_0{}^1) + H(Y_0{}^0), \text{ and } u(Y_1{}^0) = \mu u(Y_1{}^1) + H(Y_1{}^0),$$

by the definition of $DARA_R$ (See Ross Theorem 2). Now $D_1U(a_1,b)|_{W=W_0}$ can be signed as follows.

$$D_1U(a_1,b)|_{W=W_0} = (1\text{-}q)E[u'(Y_0{}^0)(R\text{-}R_f)] + q\, E[u'(Y_1{}^0)(R\text{-}R_f)]$$

$$= (1\text{-}q)\, E\{[\mu u_B{}'(Y_0{}^1)+H'(Y_0{}^0)](R\text{-}R_f)\} + q\, E\{[\mu u_B{}'(Y_1{}^1)+H'(Y_1{}^0)](R\text{-}R_f)\}$$

$$= \mu\, \{(1\text{-}q)\, E[u'(Y_0{}^1)(R\text{-}R_f)] + q\, E[u'(Y_1{}^1)(R\text{-}R_f)]\}$$

$$\quad + (1\text{-}q)\, E[H'(Y_0{}^0)(R\text{-}R_f)] + q\, E[H'(Y_1{}^0)(R\text{-}R_f)]$$

$$= \mu\, D_1U(a_1,b)|_{W=W_1} + (1\text{-}q)\, E[H'(Y_0{}^0)(R\text{-}R_f)] + q\, E[H'(Y_1{}^0)(R\text{-}R_f)]$$

$$= \{(1\text{-}q)E[H'(Y_0{}^0)]E(R\text{-}R_f) + qE[H'(Y_1{}^0)]E(R\text{-}R_f)\} \qquad (<0)$$

$$\quad + \{(1\text{-}q)\, Cov[H'(Y_0{}^0);R] + q\, Cov[H'(Y_1{}^0);R]\} \qquad (<0)$$

$$< 0. \qquad\qquad (21)$$

The first term on the RHS of the last equality in (21) is negative since $H' \leq 0$ and $E(R) > R_f$. The second term on the RHS of the last equality in (21) is also negative, since $Cov[H'(Y_0^0);R]$ or $Cov[H'(Y_1^0);R]$ is negative due to $H''<0$. Similarly, the sign of $Cov[H'(Y_1^0);R]$ can be shown as negative. Therefore, part (1) of the proposition is proved. Part (2) of the proposition directly follows from and the definition of CARA. The proof of part (3) of the proposition will be skipped, since it requires similar procedure for the proof of part (1) except that u exhibits $IARA_R$.   Q.E.D.

For a given coverage of insurance, the individual passes any changes in initial wealth value to the investment portfolio through changes in the level of risky assets. Given $DARA_R$ Arrow's (1965) result that risky investment is a superior good is still robust in the presence of property-liability risks if Arrow-Pratt characterization of risk aversion is replaced with strong one of Ross.

In the following Proposition 4, using Lemma 3 and Lemma 4, the simultaneous wealth effects on the optimal composition and size of the investment portfolio as well as on the optimal coverage of insurance can be evaluated.

Proposition 4.

For an increase in initial wealth, the following results are obtained:

(1) If u exhibits $DARA_R$ and if $D_{12}U \leq 0$, the simultaneous effects are positive on the optimal level of risky assets and negative on the optimal amount of insurance; that is,

$$\left. \begin{array}{l} DARA_R \\ D_{12}U \leq 0 \end{array} \right\} \quad \Rightarrow \quad \left\{ \begin{array}{l} \dfrac{\partial a}{\partial W} > 0 \\ \dfrac{\partial b}{\partial W} < 0 \end{array} \right. ;$$

(2) If u exhibits CARA, the simultaneous effects are null both on the optimal level óf risky assets and on the optimal coverage of insurance; that is,

$$CARA \quad \Rightarrow \quad \left\{ \begin{array}{l} \dfrac{\partial a}{\partial W} = 0 \\ \dfrac{\partial b}{\partial W} = 0 \end{array} \right. ;$$

(3) If u exhibits $IARA_R$ and if $D_{12}U \leq 0$, the simultaneous effects are negative on the optimal level of risky assets and positive on the optimal amount of insurance; that is,

$$\left. \begin{array}{c} IARA_R \\ D_{12}U \leq 0 \end{array} \right\} \quad \Rightarrow \quad \left\{ \begin{array}{c} \dfrac{\partial a}{\partial W} < 0 \\ \dfrac{\partial b}{\partial W} > 0 \end{array} \right. .$$

Proof: By totally differentiating the first order conditions (1) and (2), and solving for da/dW and db/dW by Cramer's Rule, one obtains

$$\frac{\partial a}{\partial W} = \frac{D_{12}U\ D_{2W}U - D_{1W}U\ D_{22}U}{|H|}, \tag{22}$$

$$\frac{\partial b}{\partial W} = \frac{D_{21}U\ D_{1W}U - D_{2W}U\ D_{11}U}{|H|}, \tag{23}$$

where $D_{1W}U = d^2E[u(Y)]/dadW$, and so on. Now, the sign of $\partial a/\partial W$ and $\partial b/\partial W$ for an increase in W can be evaluated by starting from the initial optimal points of a* and b*, where a* and b* satisfy the first order conditions (1) and (2), respectively. To determine the influence of an increase in initial wealth value on the investment and insurance decision, the sign of $D_{1W}U$ and $D_{2W}U$ need to be determined. First, we will consider a case of $DARA_R$. Our non-simultaneous results imply that $D_{1W}U > 0$ by Lemma 4 and $D_{2W}U < 0$ by Lemma 3. Since $D_{11}U$ and $D_{22}U$ are negative, da/dW and db/dW have the same signs $D_{1W}U$ and $D_{2W}U$ respectively, if $D_{12}U \leq 0$. This complete the proof of part (1). Under CARA, we obtained $D_{1W}U = 0$ and $D_{2W}U = 0$, which makes the right-hand side of (22) and (23) to be zero. This completes the proof of part (2). In case of $IARA_R$, our non-simultaneous results imply that $D_{1W}U < 0$ and $D_{2W}U > 0$, which make da/dW and db/dW have the same signs as $D_{1W}U$ and $D_{2W}U$ respectively if $D_{12}U \leq 0$. This completes the proof of part (3).

Q.E.D.

For CARA case, the optimal decisions are totally insensitive to changes in the value of initial wealth (or the degree of risk aversion). Under $DARA_R$ insurance is possibly an inferior good, but they do not generally hold in the present model. However, as implied by our previous discussion, the investment and insurance decisions are interrelated. This interaction raise the possibility that simultaneous effect may modify the results of Mossin (1968). Thus, the results of our model are substantially different from those of Mossin (1968) in the sense that insurance may not be regarded as an inferior good even if $DARA_R$ is

assumed. The different results are a consequence of differences in the models. In Mossin (1968)'s, the only decision variable is the amount of insurance, and the simultaneous investment opportunity is never considered, whereas both decisions are simultaneously considered in the model. The results of our model include those of Mossin (1968) as special cases. That is, when the insurance decision can be made after the investment decision, the insurance can be regarded as an inferior good under $DARA_R$, as shown in Lemma 3. When an individual can make simultaneous decisions, the insurance can be regarded as an inferior good under $DARA_R$ if risky investments and insurance are independent ($D_{12}U=0$) or work like the stochastic substitutes each other ($D_{12}U<0$), as shown in Proposition 4. Similarly, the result of Arrow (1965) that risky investment is a superior good are valid only under restricted conditions.

7. Summary and Conclusions

Recent financial theory views insurance policies as financial instruments that are traded in markets and whose prices reflect the forces of supply and demand. If the individual treats insurance contracts as financial assets along with risky and riskfree financial assets in determining their portfolio, the insurance purchase decision must be explicitly considered in the individual's budget constraint. However, the joint treatment of insurance and capital market investment has received little attention within a general expected utility framework. In this paper, we have presented a financial model which considers the simultaneous choice of both insurance and capital market decisions.

Theory and evidence have shown that the stochastic relationships between insurable risk and the capital market risk is not correlated, so that assumption was adopted here. After model was developed, the aspects of optimal insurance condition as well as optimal investment portfolio was analyzed to demonstrate that optimal insurance purchasing strategies critically depend on the capital market investment opportunities. moreover we have shown that how the introduction of a capital market investment could affect the four well-known traditional theorems on insurance. The results of this paper supplements the traditional theories on insurance and suggest some extensions. The main results can be summarized as follows.

(1)   The individual will select its level of insurance so that the insurance premium equals the expected value of losses plus the positive loading. The size of insurance loading, which

the individual is willing to pay in addition to the actuarial value of insurance policy, is not only a function of the riskiness of insurable losses and risk aversion but also a function of the capital market investment returns. Thus, the equilibrium premium reflects the capital market income which the individual recognize in investing financial assets. This result clearly shows that the financial-economic concept of insurance cost should consider the opportunity cost of insurance premium.

(2)   It will never be optimal for the individual exposed to insurable risks to share his risks only with insurer without being involved in risky stock. That is, if capital market exists, every individual invests in risky stock and simultaneously adjust the coverage of insurance according to his characterization of risk aversion. For example, if decreasing absolute risk aversion is a reasonable assumption, the existence of capital market will induce the individual to demand less insurance than it would have in the absence of such a market.

(3)   If insurable losses and capital market returns are stochastically characterized by non-interdependence which is defined on joint distribution function, as assumed in the present model throughout the paper, rational individual's behavior implies purchasing full insurance coverage if the premium is fair or that full coverage is suboptimal if the premium is unfair. The results are interpreted by mean preserving spread of Rothschild and Stiglitz (1970).

(4)   An increase in the absolute risk aversion in Ross sense reduces the size of risky assets in the optimal investment portfolio and increases the optimal amount of insurance, if risky investments and insurance interact as stochastic substitutes for each other or if risky investments and insurance are independent.

For an increase in initial wealth, depending upon whether absolute risk aversion in Ross sense is a decreasing, constant, or increasing function of final wealth, the simultaneous effects are positive, null, or negative on the level of risky assets in the optimal investment portfolio, and negative, null or positive on the optimal amount of insurance, assuming risky investments and insurance are independent or work like stochastic substitutes for each other.

These results can be contrasted to those of traditional literature. When the investment and insurance decisions are simultaneous, interaction between insurance and investment

decision may overturn the traditional results. The traditional results holds, if risky investments and insurance interact as stochastic substitutes for each other to improve the individual's expected utility, or if risky investments and insurance are independent.

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