The Journal of Natural Science,
Pai Chai University, Korea
Volume 5(1) : 1－ 3, 1992

CLASSIFICATION OF THE EQUIVARIANT LINE BUNDLES OVER S^1**

SUNG SOOK KIM*

원 위에서의 EQUIVARIANT LINE BUNDLE 의 분류**

김 성 숙*

개요: G 가 compact Lie 군이고 $\pi : E \to S^1$ 이 S^1 상의 G-line bundle 일때, 군 작용이 없다면, 부드러운 trivial G-line bundle $E \to S^1$ 은 $S(V) \times \delta \to S(V)$ 와 동치이고 부드러운 nontrivial G-line bundle $E \to S^1$ 은 $S(V) \times_{\mathbf{Z}_2} \delta \to S(V)/\mathbf{Z}_2 = P(V)$ 와 동치 이다.

1. Introduction.

Let G be a compact Lie group and let S^1 denote the unit circle in R^2. In this paper, we classify equivariant line bundles over S^1. Let $\pi : E \to S^1$ be a G line bundle over S^1. We assume the G-action on E to be effective. Since any smooth G action on S^1 is smoothly equivariant to a linear action [Sc Theorem 2.0], we may assume the G action on the base space S^1 is linear. So the G action on S^1 gives a homormorphism $\rho : G \to O(2)$.

Lemma 1. *Ker ρ is trivial or of order 2. In particular ker ρ is in the center of G.*

Proof: By the definition of ρ, ker ρ acts on the base space S^1 trivially. So it should act non trivially on each fiber because our action on the total space E is effective. Since the fiber is $\mathbb{R}$ and the action of ker ρ on $\mathbb{R}$ is linear, ker ρ must be trivial or of order 2.

---

* 배재 대학교 이공 대학 응용 수학과 조교수.

** 이 논문은 한국 과학 재단 연구비 913-0103-015-2 에 의해 연구 된 것 의 일부 이기에 이에 감사 드립니다.

한글 $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX 에 의한 전자조판

Because $\rho(G)$ is a subgroup of $O(2)$, it is a cyclc group, a dihedral group, $SO(2)$ or $O(2)$. Hence G is decided when ker ρ is trivial. When ker ρ is of order 2 there are several choices of G with same $\rho(G)$. But also we get a further restriction on G which comes from the effectiveness of the action. We study two cases according to $\pi : E \to S^1$ being trivial bundle or non trivial bundle when we forget the action.

Acknowledgments. I would like to thank Professor Mikiya Masuda for his encouragement and helpful conversations.

2. Nonequivariantly trivial bundle case.

Suppose $\pi : E \to S^1$ is trivial when we forget the action. Since G is compact $\pi : E \to S^1$ with a fiber metric is isomorphic to the trivial line bundle $S^1 \times \mathbf{R} \to S^1$ with the standard fiber metric.

Hence one can express the action of $g \in G$ on $S^1 \times \mathbf{R}$ as

$$(x, v) \to (\rho(g)(x), \varphi_g(x)v) \qquad \text{for } (x, v) \in S^1 \times \mathbf{R}$$

where S^1 is viewed as the unit circle in $\mathbf{R}^2$, $\rho : G \to O(2)$ is a homomorphism, $\rho(g)$ acts on S^1 in the standard way and $\varphi_g(x)$ is a scalar. Since the action of g preserves the standard metric on $S^1 \times \mathbf{R}$, $\varphi_g(x)$ must be ± 1. The map $\varphi_g : S^1 \to \{\pm 1\} = \mathbf{Z}_2$ is continuous and S^1 is connected. So we have the following Lemma.

Lemma 2. *For a fixed $g \in G, \varphi_g(x)$ is independent of $x \in S^1$, i.e. $\varphi_g(x) = 1$ for all x or $\varphi_g(x) = -1$ for all x.*

By virtue of Lemma 2, we have a homorphism $\varphi : G \to \mathbf{Z}_2 = \pm 1$ given by $g \to \varphi_g$. Hence $\varphi_g = 1$ if g is of odd order, in particular φ is trivial when G is of odd order or $SO(2)$.

Now we show which group G can act on line bundles E over S^1 effectively.

Lemma 3. *Suppose $\pi : E \to S^1$ is nonequivariantly trivial line bundle. Then G is isomorphic to $\rho(G) \times ker\rho$.*

Proof: Since G is compact, E admits a G invariant fiber metric. Hence the total space of the sphere bundle of E, denoted by $S(E)$, is invariant under the action of G when we forget the action. The sphere bundle $S(E)$ of E is diffeomorphic to $S^1 \times S^0$ because E is trivial. Now if Ker ρ is trivial, then the lemma is obvious so we assume kerρ is of order 2. let G_+ be the subgroup of G which consists of elements preserving the connected components of $S(E)$. Since the non-trivial element of kerρ acts as multiplication by -1 on each fiber of E, it interchanges the connected components of $S(E)$. Hence the multiplication $G_+ \times ker\rho \to G$ gives an isomorphism. Because ρ gives an isomorphism: $G_+ \to \rho(G)$ this proves the lemma.

3. Nonequivariantly nontrivial bundle case.

Lemma 4. *Suppose $\pi : E \to S^1$ is non-trivial when we forget the action. Then G is a subgroup of $O(2)$.*

Proof: In this case the total space of the sphere bundle of E, denoted by $S(E)$, is diffeomorphic to S^1. Since the action of G on $S(E)$ is effective and smoothly equivariant to a linear action, G is a subgroup of $O(2)$.

The lemma above shows which group G can act on line bundles E over S^1 effectively. Next we must study how G acts on E. Suppose $\pi : E \to S^1$ is nontrivial when we forget the action. Then the projection $\pi : S(E) \to S^1$ is an equivariant double covering map where G acts on the base space S^1 through the homormorphism ρ. The induced G-line bundle by π from E is trivial, so the problem to decide action of G on the non-trivial line bundle overS^1 is reduced to the previous case. Now we obtain the following Theorem.

4. Main result.

Main Theorem. *A smooth G-line bundle $E \to S^1$ is equivariantly isomorphic to $S(V) \times \delta \to S(V)$ or $S(V) \times_{\mathbf{Z}_2} \delta \to S(V)/\mathbf{Z}_2 = P(V)$ according as $E \to S^1$ is trivial or not when we forget the action. Here $S(V)$ denotes the unit circle of a real 2-dimensional orthogonal G-module V, δ a real 1-dimensional G-module and $\mathbf{Z}_2$ acts on $S(V)$ and δ as scalar multiplication.*

참 고 문 헌

[B] G.E. Bredon, *Introduction to Compact Transformation Groups*, Pure and Applied Math. Vol. 46, Academic Press, 1972.

[H] W.Y. Hsiang, *Cohomology Theory of Topological Transformation Groups*, Ergebnisse der Math. Band 85, Springer-Verlag, 1975.

[Sc] R. Schultz, *Nonlinear analogs of linear group actions on spheres*, Bull. AMS (New series) 11 (1984), 263–285.

[S] R.E. Stong, *Unoriented Bordism and Actions of Finite Groups*, Memoirs of the AMS 103, 1970.