Journal of the KSQC
Vol. 20, No. 1, June. 1992

Comparisons of Two-Stage Acceptance Life Test Sampling Plans for Exponential Lifetime Distribution

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Abstract

This thesis compares life test acceptance sampling plans under lifetime has an exponential distribution. Various practical considerations may lead a user adopt a two-stage, or double sampling, test procedure. Hewett and Spurrier(1983) provided a survey of two-stage methods, as well as examples of experiments for which a two-stage procedure would be appropriate.

The plans are compared in terms of the expected number of failures, and the expected time required to reach a dicision. Computational experiments are conducted and the results are tabulated to provide guidelines for selecting an appropriate plan for a given situation

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1. Introductiom

For a number of types of manufactured products, components and materials lifetime may be one of the quality caracteristics specified by the designer. An acceptance sampling plans for life times should serve the dual purpose of,

1) weeding out lots which are unsatisfactory quality, thereby improving the overall quality,

2) exerting pressure on the producer to keep the manufacturing process in control.

Factors such as the high cost of evaluating a sampled item and physical limitations on the number of items that may be tested at any one tim,e could motivate the use of a two-stage life testing. Such two-stage life testing gives "second chance" to be accepted.

When the purpose of the life test is for reliability acceptance sampling , we compare the two censoring methods in terms of power, expected number of failures, and expected completion time. We found that the two censoring methods are similar in term of power. If the true mean lifetime is close to the value specified by H_0, then type I censoring is preferred to type II censoring in term of the expected number of failures and expected completion time. If the true mean lifetime is close to the value specified by H_1, the reverse is true.

Epstein(1954) proposed a test combining both types of censoring, sometimes called hybrid censoring. Although there is an expected time savings in using type II censoring, the procedure is open-ended with respected to the time duration of the test. In adding type I censoring, the user gains the practical advantage of being able to preassign a stop time for the test. MIL-STD-781C(1977) used hybrid censoring in reliability acceptance tests.

Notations

In this paper the following notations are used.

θ	: Mean lifetime, or MTBF of an exponential distribution.
θ_0	: Specified mean lifetime under H_0.
θ_1	: Specified mean lifetime under H_1.
θ_2	: Indifference level
α	: Size of the test, or alternatively, producer's risk.
β	: Consumer's risk.
d_i	: discrimination ratio($=\theta_0/\theta_i$, $i=0, 1, 2$)
n	: Total number of items placed on test.
k_i	: Acceptable constant or critical value, $i=0, 1, 2, 3$.
r	: Number of failures.

T_i : Truncation time under type i censoring
$\chi^2(v, \alpha)$: The αth quantile of a chi-square distribution with v degrees of freedom.
$\Gamma(k)$: Gamma function with k.
$OC(d_i)$: Acceptance prob. of items under d_i
$E[K \mid d_i]$: Expected number of failures under d_i
$E[T \mid d_i]$: Expected test time under d_i
P_{ia} : Acceptable probability of lots at ith stage.
P_{ic} : Probability of the test continue to the 2nd stage.
P_{ir} : Rejectable probability of lots at ith stage.

Assumptions

1) The lifetime distribution of a test unit follows an exponential dist.
2) The lifetimes of test units are statistically independent.
3) The items in lots are produced by a stable manufacturing process.
4) Failed items are not replaced by new items.

Overview of Related Literatures

The literature in reliability theory contains many testing and estimation prorcedures involving type I (time) censoring or type II (failure) censoring. These procedures often include an assumption that the lifetime in question has an exponential distribution. Mann, Schafer and Singpurwalla(1974), Bain(1978), Fairbanks (1988) and others have reviewed the theory and methods of these procedures. Also Military Standard 781C(1977) used hybrid censoring in reliability acceptance tests. Especially Heweet(1973, 1983) provided a survey of two-stage methods under type II censoring and Fairbanks(1988) compared sampling plan under two-stage hybrid censoring to in case of one-stage.

Table 1. Classification of Life Test Sampling Plans

Authors	Type of sampling	Type of censoring	Sampling plan	Decision criterion	Remarks
Epstein ('53)	one-stage	type-II	(r, n) r : number of failures	$\theta r, n > C$ Accept. H_0 $H_0 : \theta=\theta_0$ $H_1 : \theta=\theta_1$	$2r\theta r, n \sim \chi^2(2r)$ $C=\theta_0\ \chi^2(2r)/2r$
Epstein ('60)	one-stage	hybrid	Min. (Xr, T)	Xr, n>T Accept. H E[R], E[T]	

Bulgren/ Hewett ('73)	two-stage	type-II	$(r_1, r_2, t_1, t_2, t_3)$	if $Xr \le t_1$, Accept. H_0, if $t_1 < Xr < t_2$, continue. if $Xr_1 + r_2 \le t_3$, accept. H_0	$r_1 = 0.4r_0$ $r_2 = 0.8r_0$
Angus et. ('85)	one-stage	type-I	(t, T) $0 < t < T$	$wt \le T$ Accept H_0 $H: \lambda = \lambda_0$, $H: \lambda = \lambda_1$ $\lambda_1 > \lambda_0$	t=FFP. T ; test time Wt ; waiting time at t
Ohta/Arizono ('86)	one-stage	type-I	$I(g : gi)$	$I(g : g_0) \le I(g : g_1)$, accept the lot.	$I(g : g_i)$: Kullback Leibler Information Informat-
Fairbanks ('88)	two-stage	hybrid	$(r_1, r_2, T_1, T_2, T_3)$ r_i : number of failures T : test time	Xr_1, $n \ge T_1$ accept H_0 Xr_2, $n < T_3$ reject H_0 $T_1 \le Xr$, $n < T_2$ continues	

2. Two-Stage Life Test Procedures

1. type I censoring

Two-stage life test sampling plan under I censoring on MIL-STD-781C for fixed-length life test sampling plans, i. e., to match three points of OC curve. MIL-STD-781C for fixed-length life tests are available only when the batch of items has life length that are assumed to have an exponential distribution.

The proposed double life test under type I censoring provides an opportunity for an early decision while maintaining the desired significance level. The test time is multiples of MTBF(θ_0).

The objective is to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1 (\theta_0 > \theta_1)$,

$$f(x ; \theta) = (1/\theta) \exp(-x/\theta), \quad x \ge 0$$
$$= , \qquad x < 0.$$

1st stage : terminate testing at T_1,

$r_1 \le k_1$; test would accept H_0, $r_1 > k_1$; test would reject h_0.
$k_1 < r_1 \le k_2$; test continues to the 2nd stage.

2nd stage : terminate testing at $T_2(T_2>T_1)$,

$r_1+r_2 \leq k_3$; test would accept H_0, $r_1+r_2>k_3$; test would reject H_0.

So an acceptance criterian is the value which guarantees the specified mean time between failure(MTBF) with the producer's and/or consumer's risk to minimized the expected number of failures at θ_0.

$$\underset{T_1,\ T_2,\ k_1,\ k_2,\ k_3}{\text{Min.}} \{E[R_0 \mid \theta_0] = E[R_0 \mid d_0]\}$$

$$= \{1-P_{1c}(d_0)\}\ T_1 + P_{1c}(d_0)\ T_2 \qquad (2-1)$$

subject to

$$OC(d_0)=1-\alpha$$
$$OC(d_1)=\beta$$
$$OC(d_2)=0.5$$

With the lot on test, the user preassingns times T_1, T_2 and determines fixed number of failures k_1, k_2 and k_3. Since k_i is integer valued, $k_1<k_2 \leq k_3$, $0<T_1<T_2$, it has been possible to locate approximate solutions of three nonlinear equations using an interactive computer grid search method.

① $P_{1a}(d_i)$

$$= \sum_{x=0}^{k_1} \frac{1}{x!} \{\exp(-d_iT_1)(d_iT_1)^x\}$$

$$= 1 - \int_0^{T_1} \frac{(d_i)^{k+1} \cdot x^k \cdot \exp(-d_ix)}{\Gamma(k_1+1)} dx$$

$$= 1 - \int_0^{2d_iT_1} f(z)dz$$

② $P_{1r}(d_i)$

$$= \sum_{x=k2+1}^{k_1} \frac{1}{x!} \{\exp(-d_iT_1)(d_iT_1)^x\}$$

$$\int_0^{T_1} \frac{(d_i)^{k+1} \cdot x^k \cdot \exp(-d_ix)}{\Gamma(k_2+1)} dx$$

$$\int_0^{2d_iT_1} f(z)dz \qquad (=1-P_{1a}(d_i)-P_{1c}(d_i))$$

③ $P_{1c}(d_i)$

$$= \sum_{x=k1+1}^{k2} \frac{1}{x!} \{\exp(-d_iT_1)(d_iT_1)^x\}$$

$$= 1 - \int_0^{T_1} \frac{(d_i)^{k+1} \cdot x^k \cdot \exp(-d_ix)}{\Gamma(k_1+1)} dx - P_{1a}(d_i)$$

$$= 1 - \int_0^{2d_iT_1} f(z)dz - P_{1a}(d_i)$$

$$= 1 - \int_0^{2d_iT_1} f(z)dz - \int_0^{2d_iT_1} f(z)dz$$

④ $P_{2a}(d_i)$

$$= \sum_{y=0}^{k3-r1} \frac{1}{y!} \{\exp(-d_iT_2-T_1)(d_iT_i)^y\},$$

$$= 1 - \int_0^{T_2-T_1} \frac{(d_i)^{k-r+1} \cdot x^{k-r} \cdot \exp(-d_ix)}{\Gamma(k_3-r_1+1)} dx$$

$$= 1 - \int_0^{2(T_2-T_1)d_i} f(z)dz$$

⑤ $P_{2r}(d_i) = 1 - P_{2a}(d_i)$

Assuming that the user specifies α and β for some chosen $\theta_1 \langle \theta_0$. I have chosen to match the OC curves at probability $OC(d_0) = 1-\alpha$, $OC(d_1) = \beta$ and $OC(d_2) = 0.5$. These valuse should be widely spaced and the value d_i, corresponding to these probabilities on MIL-STD-781C's OC curve, are easily found. Hence, $OC(d_i) = P_{1a}(d_i) + P_{1r}(d_i) \cdot P_{2a}(d_i)$.

$$OC(d_0) = P_{1a}(d_0) + P_{1r}(d_0) \cdot P_{2a}(d_0)$$

$$= \{1 - \int_0^{2T_1} f(z_1)\ dz_1\} + \{\int_0^{2T_1} f(z_2)\ dz_2 - \int_0^{2T_1} f(z_1)\ dz_1\}$$

$$\times \{1 - \int_0^{2(T_2-T_1)} f(z_3)\ dz_3\} = 1-\alpha \qquad (2-2)$$

$$OC(d_1) = \{1 - \int_0^{2d_1T_1} f(z_1)\ dz_1\} + \{\int_0^{2d_1T_1} f(z_2)\ dz_2 - \int_0^{2d_1T_1} f(z_1)\ dz_1\}$$

$$\times \{1 - \int_0^{2d_1(T_2-T_1)} f(z_3)\ dz_3\} = \beta \qquad (2-3)$$

$$OC(d_2) = \{1 - \int_0^{2d_2T_1} f(z_1)\ dz_1\} + \{\int_0^{2d_2T_1} f(z_2)\ dz_2 - \int_0^{2d_2T_1} f(z_1)\ dz_1\}$$

$$\times\{1-\int_0^{2d_2(T_2-T_1)} f(z_3)\ dz_3\}=0.5 \qquad (2-4)$$

where $z_1=\chi^2(2k_1+2,\ 1-\alpha)$, $z_2=\chi^2(2k_2+2,\ \beta)$ and $z_3=\chi^2(2k_3-2r_1+2,\ 0.5)$, $k_1<r_1\le k_2$.

since n is integer valued and $0<T_1<T_2$, it is possible to locate approximate solutions to this theory of three nonlinear equations using an interactive computer grid search.

Algorithm

Step 1) given α, β(from MIL-STD-781C)

1) Calculation K_0, T_0

Determine k_0 which satisfied $\chi^2(2k_0+2,\ \alpha)/\chi^2(2k_0+2,\ 1-\beta)\ge 1/d_1$, therefore input k_0 to following as ;

$$OC(d_0)=1-\int_0^{2T_0} f(z_0)\ dz_0,\ \text{where } z_0=\chi^2(2k_0+2,\ 1-\alpha) \qquad (2-5)$$

2) Calculation d_2

By using the equation (2-4), We find $OC(d_2)$ at $d_2=\theta_0/\theta_1$,

$$\chi^2(2k_0+2,\ 0.5)=2\ T_0\ d_2$$
$$d_2=\chi^2(2k_0+2,\ 0.5)/2T_0$$

3) Calculation $E[R_0 \mid d_i]$

$$P(r\ ;\ d_i)=1/r!\ \{\exp(-T_0d_i)(T_0d_i)^2\},\ r=1,\ 2,\ \cdots,\ k_0$$

$$=1-\sum_{r=1}^{k_0} 1/r!\ \setminus\{\exp(-T_0d_i)(T_0d_i)2\},\ r=k_0+1$$

$$E[R_0 \mid d_i]=\sum_{r=1}^{k_0+1} r\cdot P(r)$$

4) Calculation $E[T\cdot\theta_0 \mid \theta_i]$

$$E[T \cdot \theta_0 \mid \theta_i] = \theta_0 \; E[T \mid \theta_i]$$
$$= \theta_0 \cdot E[K_0 \mid d_i] / d_i$$

Step 2) Calculation T_1, T_2, given T_0

Since T_0 is determined from MIL-STD-781C, T_1 and T_2 should determine according to the guidelines of Zeigler and Tietjen(1968). They suggested choosing r_1 slightly less than or slightly greater than $0.5r$, where r is the number of faiures to be observed in the corresponding single-stage test and r_2 such that $r_1 + r_2$ is slightly greater than r(e.g., $r_1 + r_2 = 1.5r$).

$T_1 = 0.26\ T_0 \sim 0.75\ T_0$
$T_2 = 1.01\ T_0 \sim 1.50\ T_0$

Step 3) Calculation k_1, k_2 and k_3 from $OC(d_i)$

Determine the smallest integer combination (k_1, k_2, k_3) satisfied following as ;

$OC(d_0) \fallingdotseq 1 - \alpha$

Step 4) Using the grid search methods, determine(T_1, T_2, k_1, k_2, k_3) which minimize the expected number of failures. With known values for (T_1, T_2, k_1, k_2, k_3), we can find an approximate solution of three nonlinear equation, (2-2), (2-3), (2-4). The grid boundaries and the size of the grid are reduced interactively until we achive convergence to a desired level of precision.

3. Numerical Experiment

As an example of implementing the test procedure, we consider a production reliability acceptance test of a electric items. The consumer will assept a production lot with $1-\alpha=0.90$ if the true MTBF $\theta_0 = 100$ hours and will reject the lot with $\beta=0.2$ if $\theta_1 = 50$.

In this case, $T_1 = 3.78$, $T_2 = 8.00$ and $K_1 = 4$, $k_2 = 9$, $k_3 = 11$.

1st Stage : test de termined at $T_1 = 378$ ($3.78 * 100 = 378$).

$r_1 \le 4$; test would accept, $r_1 > 9$: test would reject.
$4 < r_1 \le 9$; test continue to the 2nd stage.

2nd Stage : test de terminted at $T_2=422$.

$r_1 + r_2 \leq 11$: test would accept, $r_1 + r_2$ 11 ; test would rejuct.

(total time on test = 378 + 422 = 800 hours)

Table 2. The results of two-stage sampling plan

α (%)	β (%)	d_1	d_2	T_1, T_2	k_1, k_2, k_3	α '	β '
10	10	1.5	1.22	12.3, 36. 9	9, 24, 44	10. 5	7.48
10	20	1.5	1.29	6.96, 23.88	4, 15, 30	8.99	19. 6
20	20	1.5	1.26	4.51, 17.28	3, 12, 19	20. 1	19. 5
10	10	2.0	1.45	4.14, 10.38	3, 10, 14	9.61	10. 0
10	20	2.0	1.56	3.78, 8.00	4, 9, 11	9.14	19. 3
20	20	2.0	1.45	2.53, 4.76	2, 4, 6	20. 6	19. 8
10	10	3.0	1.83	1.98, 3.75	2, 4, 6	9.93	9.85
10	20	3.0	2.04	1.12, 2.63	1, 3, 4	10. 0	19. 9
20	20	3.0	1.87	0.78, 2.09	0, 1, 4	20. 1	19. 6

(α ', β ', : true value)

Table 3. Comparisons with MIL-STD-781C

α (%)	β (%)	d_1	E [K | d_0]	
			MIL-STD-781C	Double Samp.
10	10	1.5	29.65597	27.79032
10	20	1.5	19.67174	15.30877
20	20	1.5	13.75000	10.92099
10	10	2.0	9.28206	7.79016
10	20	2.0	6.10446	5.14390
20	20	2.0	3.72528	3.31657
10	10	3.0	3.04039	2.52367
10	20	3.0	1.74995	1.53920
20	20	3.0	1.35297	1.25006

4. Conclusion

This paper compared the double life test sampling plans to match three points of OC curve.

MIL-STD-781C for fixed-length life tests are applicable only when the batch (lot) of items has life length that are assumed to have an exponential distribution. Since k_i is integer valued $k_1 < k_2 \leq k_3$, $0 < T_1 < T_2$, it has been possible to locate approximate solutions of three nonliner equations using an interactive computer grid search.

Apparently, a criterion for selecting failure sizes and an efficient method for solving the resulting nonlinear equations would be the main problems in such an extention. In the results of comparison with MIL-STD-781c for fixed-length life tests to have almost identical OC curve, double life test plans yield smaller than the expected number of failures of corresponding MIL-STD-781C life test plans. In Search of the limited applications of the exponential distribution, perhaps a more useful extention would be a two-stage test for the more common Weibull distribution.

References

1. Mann, N.R., Schafer, R.E. and Singpurwalla, N.D.(1974), "Method for Statistical Analysis of Reliability and Life Data", John Wiley and Sons, New York.
2. Lawless, J.F.(1982), "Statistical Models and Methods for Lifetime Data", John Wiley and Sons, New York.
3. MIL-STD-781C,(1977), "Reliability Design Qualification and Production Acceptance Tsts, Exponential Distribution", U.S. Government Printing Office, Washington, D.C..
4. Epstein, B. and Sobel, M.(1953), "Life Testing", Journal of the American Statistical Association, 48, pp. 485-502.
5. Epstein, B.(1960), "Statistical Life Test Acceptance Procedures", Technometrics, 2, pp. 435-446.
6. Zeigler, R.K. and Tietjen, G.L.(1968), "Double Sampling Plans where the Acceptance Criterion is the Variance", Technometrics, 10, pp. 99-106.
7. Bulgren, W. and Heweet, J.(1973), "Double Sample Tests for Hypotheses about the Mean of an Exponential Distribution", Technometrics, 15, pp. 187-190.
8. Hewett, J. and Spurrier, J.(1983), "A Survey of Two-Stage Tests of Hpotheses : Theory and Application", Communications in Statistics-Theory and Methods, 11, pp. 2307-2425.
9. Kocherlakota, S. and Balakrishnan, N.(1986), "One-and Two-Sided Sampling Plans based on the Exponential Distribution", Naval Research Logistics Quarterly, 33, pp. 513-522.
10. Arizono, I. and Ohta, H.(1986), "A Simplified Design-Procedure for Fixed-time Life Tests Based on Kullback-Leibler Information", IEEE Transaction Reliability, R-35, No. 3, pp. 579-580.
11. Ebrahimi, N.(1988), "Determining the sample size for a Hybrid Life Test Based on the Cost Function", Naval Research Logistics Quaterly, 35, pp. 63-72.
12. Fairbanks, K.(1988), "A Two-Stage Life Test for the Exponential Parameter", Technometrics, 30, pp. 175-180.
13. Griffiths, P. and Hill, I.D.(1985), "Applied Statistics Algorithms", The Royal Statistical Society, London.