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ON THE EXTENDED GOTTLIEB SUBGROUP

Moo Ha Woo[1] and Ki Young Lee[2]

Dedicated to Professor Younki Chae on his 60th birthday

1.Introduction

F. Rhodes [2] introduced the fundamental group $\sigma(X, x_0, G)$ of a transformation group (X, G) as a generalization of the fundamental group of a topological space X and showed a sufficient condition for $\sigma(X, x_0, G)$ to be isomorphic to $\pi_1(X, x_0) \times G$, that is, if (G, G) admits a family of preferred paths at $e, \sigma(X, x_0, G)$ is isomorphic to $\pi_1(X, x_0) \times G$. D.H. Gottlieb[1] introduced the evaluation subgroup $G(X, x_0)$ of the fundamental group of X and showed a condition to be $G(X, x_0) = Z(\pi_1(X, x_0))$. In [3], we introduced the extended Gottlieb subgroup $E(X, x_0, G)$ of the fundamental group of a transformation group (X, G). In this paper, we show a condition to be $E(X, x_0, G) = Z(\sigma(X, x_0, G))$

2. Definitions and Notations

Let (X, G, π) be a transformation group and X be a path connected compact ANR with x_0 as base point. Given an element g of G, *a path α of order g* with base point x_0 is a continuous map $\alpha : I \longrightarrow X$ such that $\alpha(0) = x_0$ and $\alpha(1) = gx_0$. A path α_1 of order g_1 and a path α_2 of order g_2 give rise to a path $\alpha_1 + g_1\alpha_2$ of order g_1g_2 defined by the equations

$$(\alpha_1 + g_1\alpha_2)(s) = \begin{cases} \alpha_1(2s), & 0 \leq s \leq 1/2 \\ g_1\alpha_2(2s-1), & 1/2 \leq s \leq 1. \end{cases}$$

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Two paths α and α' of the same order g are said to be *homotopic* if there is a continuous map $F : I^2 \longrightarrow X$ such that

$$\begin{aligned} F(s,0) &= \alpha(s) && 0 \le s \le 1, \\ F(s,1) &= \alpha'(s) && 0 \le s \le 1, \\ F(0,t) &= x_0 && 0 \le t \le 1, \\ F(1,t) &= gx_0 && 0 \le t \le 1. \end{aligned}$$

The homotopy class of a path α of order g is denoted by $[\alpha; g]$. Two homotopy classes of paths of different orders g_1 and g_2 are distinct, even if $g_1x_0 = g_2x_0$. F. Rhodes[2] showed that the set of homotopy classes of paths of prescribed order with the rule of composition $\circ$ is a group, where $\circ$ is defined by $[\alpha_1; g_1] \circ [\alpha_2; g_2] = [\alpha_1 + g_1\alpha_2; g_1g_2]$. This group was denoted by $\sigma(X, x_0, G)$, and was called the *fundamental group* of (X, G) with base point x_0.

A homotopy $H : X \times I \longrightarrow X$ is said to be an *cyclic homotopy* if $H(x,0) = x = H(x,1)$. In this case, the path $H(x_0, \cdot)$ is called the traces of the cyclic homotopy H at x_0. In [1], Gottlieb has defined $G(X, x_0) = \{[\alpha] \in \pi_1(X, x_0) | \alpha$ is homotopic to the traces of a cyclic homotopy at $x_0\}$. An equivalent definition of $G(X, x_0)$ is the following: Let $p : X^X \longrightarrow X$ be the evaluation map given by $p(g) = g(x_0)$. Then p induces a homomorphism $p_\pi : \pi_1(X^X, 1_X) \longrightarrow \pi_1(X, x_0)$. The Gottlieb subgroup $G(X, x_0)$ is the image of the homomorphism p_π. A homotopy $H : X \times I \longrightarrow X$ is said to be a *cyclic homotopy of order* g if $H(\cdot, 0) = 1_X$ and $H(\cdot, 1) = g$,where g is an element of G.

Definition 1. $E(X, x_0, G) = \{[\alpha; g] \in \sigma(X, x_0, G) | \alpha$ is homotopic to the traces of a cyclic homotopy of order g at $x_0\}$.[3]

If we define $i_G : G(X, x_0) \longrightarrow E(X, x_0, G)$ by $i_G([\alpha]) = [\alpha : e]$, then the Gottlieb subgroup $G(X, x_0)$ is identified with a subgroup of $E(X, x_0, G)$. Thus $E(X, x_0, G)$ is called the extended Gottlieb subgroup. Define $\pi' : X^X \times G \longrightarrow X^X$ by $\pi'(f, g) = gf$,then (X^X, G, π') is a transformation group and $p : (X^X, G) \longrightarrow (X, G)$ is a homomorphism. Thus p induces a homomorphism $p_\sigma : \sigma(X^X, 1_X, G) \longrightarrow \sigma(X, x_0, G)$ given by $p_\sigma([\alpha : g]) = [p\alpha : g]$. It is easy to show that $p_\sigma(\sigma(X^X, 1_X, G)) = E(X, x_0, G)$.[3]

In [2], a transformation group (X, G) is said to admit a *family of preferred paths* at x_0 if it is possible to associate with every element g of G a path k_g from gx_0 to x_0 such that the path k_e associated with the identity element e of G is homotopic to $\hat{x}_0$ and for every pair of elements g, h, the path k_{gh} from ghx_0 to x_0 is homotopic to $gk_h + k_g$, where $\hat{x}_0(t) = x_0$ for each $t \in I$.

3. An extension of the Gottlieb's results

For a group G, the center $Z(G)$ of G is defined by $\{g \in G | gh = hg$ for all $h \in G\}$. In [1], Gottlieb has shown that if X is a connected aspherical polyhedron, then $G(X, x_0) = Z(\pi_1(X, x_0))$. Now we generalize this result to $E(X, x_0, G) = Z(\sigma(X, x_0, G))$.

A family $\mathbf{K}$ of preferred paths at x_0 is called a *family of preferred traces at* x_0 if for every preferred path k_g in $\mathbf{K}, k_g\rho$ is the traces of a cyclic homotopy of order g at x_0, where $\rho(t) = 1 - t$.

Let (X, G) be a transformation group. If the transformation group (G, G) admits a family of preferred paths at e, then (X, G) admits a family of preferred traces at x_0 but the converse is not true.(see Example 2 in [3])

Lemma 1. *If k is the traces of a cyclic homotopy of order g at x_0, then for every loop α at x_0, α is homotopic to $k + g\alpha + k\rho$.*

Proof. See Lemma 2 in [3].

In [1], Gottlieb has shown that $G(X, x_0) \subset Z(\pi_1(X, x_0))$ which is a special case of the following theorem.

Theorem 2. *Let G be an abelian group. If (X, G) admits a family $\{k_g | g \in G\}$ of preferred traces at x_0, then $E(X, x_0, G) \subset Z(\sigma(X, x_0, G))$.*

Proof. Let $\mathbf{K} = \{k_g | g \in G\}$ be a family of preferred traces at x_0. For any elements $[\alpha : g_1] \in E(X, x_0, G)$ and $[\beta : g_2] \in \sigma(X, x_0, G)$, we must show $[\alpha : g_1] \circ [\beta : g_2] = [\beta : g_2] \circ [\alpha : g_1]$. Since G is abelian, it is sufficient to show that $\alpha + g_1\beta$ is homotopic to $\beta + g_2\alpha$. If we use Lemma 1 and $k_{g_1}\rho$ is the traces of a cyclic homotopy of order g_1 at x_0, we have

$$\begin{aligned}
\alpha + g_1\beta &\sim \alpha + k_{g_1} + k_{g_1}\rho + g_1\beta + k_{g_1g_2} + k_{g_1g_2}\rho \\
&\sim \alpha + k_{g_1} + k_{g_1}\rho + g_1(\beta + k_{g_2}) + k_{g_1} + k_{g_1g_2}\rho \\
&\sim \alpha + k_{g_1} + \beta + k_{g_2} + k_{g_1g_2}\rho
\end{aligned}$$

and

$$\begin{aligned}
\beta + g_2\alpha &\sim \beta + k_{g_2} + k_{g_2}\rho + g_2\alpha + k_{g_2g_1} + k_{g_2g_1}\rho \\
&\sim \beta + k_{g_2} + k_{g_2}\rho + g_2(\alpha + k_{g_1}) + k_{g_2} + k_{g_2g_1}\rho \\
&\sim \beta + k_{g_2} + \alpha + k_{g_1} + k_{g_2g_1}\rho.
\end{aligned}$$

From these results, we know that $\alpha + g_1\beta$ is homotopic to $\beta + g_2\alpha$ if and ony if $\alpha + k_{g_1} + \beta + k_{g_2}$ is homotopic to $\beta + k_{g_2} + \alpha + k_{g_1}$. Since $[\alpha : g_1] \in E(X, x_0, G)$ and $k_{g_1} \in \mathbf{K}$, there exists an cyclic homotopy H_1 of order g_1 at x_0 such that $H_1(x_0, \cdot)$ is homotopic to α and a cyclic homotopy H_2 of order g_1 at x_0 such that $H_2(x_0, \cdot)$ is homotopic to $k_{g_1}\rho$. Define $J : X \times I \longrightarrow X$ by

$$J(x,t) = \begin{cases} H_1(x, 2t), & 0 \leq t \leq 1/2 \\ H_2(x, 2-2t), & 1/2 \leq t \leq 1. \end{cases}$$

then J is a cyclic homotopy such that $J(x_0, \cdot)$ is homotopic to $\alpha + k_{g_1}$. Thus $\alpha + k_{g_1}$ is the traces of a cyclic homotopy of order 1_X. By Lemma 1, we obtain

$$\begin{aligned} \alpha + k_{g_1} + \beta + k_{g_2} &\sim (\alpha + k_{g_1}) + (\alpha + k_{g_1})\rho + (\beta + k_{g_2}) + (\alpha + k_{g_1}) \\ &\sim \beta + k_{g_2} + \alpha + k_{g_1}. \end{aligned}$$

In [1],the main result concerning the Gottlieb subgroup on a connected aspherical(in the sence that $\pi_i(X, x_0) = 0$ for $i > 1$) polyhedron is

Theorem 3[1]. *Let X be a connected aspherical polyhedron. Then $G(X, x_0) = Z(\pi_1(X, x_0))$.*

Since every transformaation group (X, G) with the trivial acting group $G = \{1_X\}$ admits a family of preferred traces at x_0, the Gottlieb's result can be extended by the following

Theorem 4. *Let X be a connected aspherical polyhedron and G be abelian. If (X, G) admits a family $\{k_g | g \in G\}$ of preferred traces at x_0, then $E(X, x_0, G) = Z(\sigma(X, x_0, G))$.*

Proof. If (X, G) admits a family $\{k_g | g \in G\}$ of preferred traces at x_0, then we first show that there exists an isomorphism ϕ from $\sigma(X, x_0, G)$ onto $\pi_1(X, x_0) \times G$ which carries $E(X, x_0, G)$ onto $G(X, x_0) \times G$. Define $\phi : \sigma(X, x_0, G) \longrightarrow \pi_1(X, x_0) \times G$ by $\phi([\alpha : g]) = ([\alpha + k_g], g)$, then ϕ is well defined. Because, if $[\alpha : g] = [\alpha' : g']$, then α is homotopic to α' and $g = g'$. Thus $\alpha + k_g$ is homotopic to $\alpha' + k_g$. Suppose $\phi([\alpha : g]) = \phi([\alpha' : g])$. Then $\alpha + k_g$ is homotopic to $\alpha' + k_g$. This implies that $\alpha (= \alpha + k_g + k_g\rho)$ is homotopic to $\alpha' (= \alpha' + k_g + k_g\rho)$. Therefore ϕ is injective.

For any element $([\alpha], g) \in \pi_1(X, x_0) \times G$, there exists an element $[\alpha + k_g\rho : g]$ in $\sigma(X, x_0, G)$ such that $\phi([\alpha + k_g\rho : g]) = ([\alpha], g)$. Therefore, ϕ is surjective.

Next, we show that ϕ is a homomorphism. Let $[\alpha_1 : g_1]$ and $[\alpha_2 : g_2]$ be elements of $\sigma(X, x_0, G)$. Then

$$\phi([\alpha_1 : g_1] \circ [\alpha_2 : g_2]) = ([\alpha_1 + g_1\alpha_2 + k_{g_1 g_2}], g_1 g_2)$$

and

$$\phi([\alpha_1 : g_1]) \circ \phi([\alpha_2 : g_2]) = ([\alpha_1 + k_{g_1} + \alpha_2 + k_{g_2}], g_1 g_2).$$

Since $\alpha_2 + k_{g_2}$ is a loop at x_0 and $k_{g_1}\rho$ is the traces of a cyclic homotopy of order g_1 at x_0, $\alpha_2 + k_{g_2}$ is homotopic to $k_{g_1}\rho + g_1(\alpha_2 + k_{g_2}) + k_{g_1}$ by Lemma 1. Therefore, we have

$$\begin{aligned}
\alpha_1 + k_{g_1} + \alpha_2 + k_{g_2} &\sim \alpha_1 + k_{g_1} + k_{g_1}\rho + g_1(\alpha_2 + k_{g_2}) + k_{g_1} \\
&\sim \alpha_1 + g_1(\alpha_2 + k_{g_2}) + k_{g_1} \\
&\sim \alpha_1 + g_1\alpha_2 + g_1 k_{g_2} + k_{g_1} \\
&\sim \alpha_1 + g_1\alpha_2 + k_{g_1 g_2}.
\end{aligned}$$

This implies that ϕ is a homomorphism. Finally, we show ϕ sends $E(X, x_0, G)$ onto $G(X, x_0) \times G$. Let $[\alpha : g]$ be an element of $E(X, x_0, G)$. Then there exists a cyclic homotopy $H : X \times I \longrightarrow X$ of order g with trace α and a cyclic homotopy $J : X \times I \longrightarrow X$ of order g with trace $k_g\rho$.

Define $F : X \times I \longrightarrow X$ by

$$F(x,t) = \begin{cases} H(x, 2t), & 0 \le t \le 1/2 \\ J(x, 2(1-t)), & 1/2 \le t \le 1. \end{cases}$$

Then F is a cyclic homotopy with trace $F(x_0, \cdot)$ which is homotopic to $\alpha + k_g$, for

$$F(x,0) = H(x,0) = x, F(x,1) = J(x,0) = x,$$

$$F(x_0,t) = \begin{cases} H(x_0, t), & 0 \le t \le 1/2 \\ J(x_0, 2(1-t)), & 1/2 \le t \le 1 \\ = (\alpha + k_g)(t). & \end{cases}$$

Thus $[\alpha + k_g]$ belongs to $G(X, x_0)$. For any element $([\alpha], g) \in G(X, x_0) \times G$, there exists a cyclic homotopy $H : X \times I \longrightarrow X$ of order 1_X with trace α. Since $\{k_g | g \in G\}$ is a family of preferred traces at x_0, there exists a cyclic homotopy $J : X \times I \longrightarrow X$ of order g with trace $k_g\rho$. Define

$$F(x,t) = \begin{cases} H(x, 2t), & 0 \le t \le 1/2 \\ J(x, 2t-1), & 1/2 \le t \le 1, \end{cases}$$

then F is a cyclic homotopy of order g with trace $\alpha + k_g\rho$ and hence there exists an element $[\alpha + k_g\rho : g]$ in $E(X, x_0, G)$ such that $\phi([\alpha + k_g\rho : g]) = ([\alpha + k_g\rho + k_g], g) = ([\alpha], g)$.

Let $[\alpha : g]$ be any element of $Z(\sigma(X, x_0, G))$. Then $\phi([\alpha : g]) = ([\alpha + k_g], g)$ belongs to $Z(\pi_1(X, x_0) \times G) = Z(\pi_1(X, x_0)) \times G = G(X, x_0) \times G$. Thus $[\alpha : g]$ belongs to $E(X, x_0, G)$. The other implication is followed by Theorem 2. Thus this completes the proof.

Remark. We can think the Gottlieb's result(that is,Theorem 3) is a special case of Theorem 5 by taking $G = \{1_X\}$.

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Department of Mathematics Education, Korea University, Seoul 136-701, Korea.

Department of Mathematics, Taejon National University of Technology, Taejon 300-170, Korea.