Kyungpook Mathematical Journal
Volume 32, Number 3, October, 1992.

T-FUZZY INTEGRAL

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Dedicated to Professor Younki Chae on his 60th birthday

In this paper, we will establish the definition of fuzzy integral with which the main properties of Sugeno's fuzzy integral [1] are retained. Then, using this definition, we show that the monotone convergence theorem and Fatou's lemma hold for the fuzzy integral.

Now, by using the T-fuzzy measure [2] and the fuzzy-valued T-fuzzy measure [3], we define T-fuzzy integrals which are very similar to Lebesgue integrals.

Definition 1. Let (X, σ) be a T-fuzzy measurable space and m a T-fuzzy measure on (X, σ). Then the T-fuzzy integral of a T-fuzzy measurable set μ with respect to m is defined as follows:

$$\int \mu dm = \sup_{\alpha \in I} \min[\alpha, m(T(\mu, \alpha))].$$

Similarly, let $\tilde{m}$ be a fuzzy-valued T-fuzzy measure on (X, σ). Then the T-fuzzy integral of a T-fuzzy measurable set μ with respect to $\tilde{m}$ is defined as follows:

$$\int \mu d\tilde{m} = \sup_{\alpha \in I} \min[\alpha, [\tilde{m}(T(\mu, \alpha))]^q].$$

By definition, we obtain the monotonicity of the T-fuzzy integral.

Proposition 2. *Let (X, σ) be a T-fuzzy measurable space and m a T-fuzzy measure on (X, σ). If $\mu_1 \leq \mu_2$ in σ, then $\int \mu_1 dm \leq \int \mu_2 dm$.*

Received March 2, 1992.

Now, we show the monotone convergence theorem and the Fatou's lemma for the T-fuzzy integral.

Definition 3. A T-fuzzy measurable space (X,σ) is said to be continuous if $(T(\mu_n,\alpha))_{n\in N} \uparrow T(\mu,\alpha)$ for all $\alpha \in I$, whenever $(\mu_n)_{n\in N} \uparrow \mu$ in σ.

Theorem 4. *Let (X,σ) be a continuous T-fuzzy measurable space and m a T-fuzzy measure on (X,σ). If $(\mu_n)_{n\in N} \uparrow \mu$ in σ, then $\lim_{n\to\infty} \int \mu_n dm = \int \mu dm$.*

Proof. Let $\lim_{n\to\infty} \int \mu_n dm = a$. Obviously, $\int \mu_n dm \leq \int \mu dm$ by the monotonicity of the T-fuzzy integral. Hence $a = \lim_{n\to\infty} \int \mu_n dm \leq \int \mu dm$. Now, suppose $a < \int \mu dm$. Take real r with $a < r < \int \mu dm$, then $\min[\alpha_0, m(T(\mu,\alpha_0))] \geq r$ for some $\alpha_0 \in I$. Therefore, $\min[\alpha_0, m(T(\mu,\alpha_0))] \geq r > a \geq \min[\alpha_0, m(T(\mu_n,\alpha_0))]$ for all $n \in N$, which is a contradiction by the fact that $(m(T(\mu_n,\alpha_0)))_{n\in N} \uparrow m(T(\mu,\alpha_0))$.

Theorem 5. *Let (X,σ) be a continuous T-fuzzy measurable space and m a T-fuzzy measure on (X,σ). Suppose σ is closed under the formations of countable* sup *and countable* inf. *Then for any sequence $(\mu_n)_{n\in N}$ in σ,*

$$\int (\lim_{n\to\infty} \inf \mu_n) dm \leq \lim_{n\to\infty} \inf \int \mu_n dm.$$

Proof. For each $n \in N$, let $\mu_n{}^* = \inf_{k\geq n} \mu_k$. Then $\mu_n{}^* \leq \mu^*{}_{n+1}$ for each $n \in N$. Since $(\mu_n{}^*)_{n\in N} \uparrow \lim_{n\to\infty} \inf \mu_n$ in σ, by Theorem 4,

$$\begin{aligned} \int \lim_{n\to\infty} \inf \mu_n dm &= \lim_{n\to\infty} \int \mu_n{}^* dm = \sup_n \int \mu_n{}^* dm \leq \sup_n \inf_{k\geq n} \int \mu_k dm \\ &= \lim_{n\to\infty} \inf \int \mu_n dm. \end{aligned}$$

Now, we obtain generalized versions of the monotone convergence theorem and the Fatou's lemma for the T-fuzzy integral.

Definition 6. A T-fuzzy measurable space (X,σ) is said to be S-continuous if for any increasing sequence $(\mu_n)_{n\in N}$ in σ, $(T(\mu_n,\alpha))_{n\in N} \uparrow T(S_{n\in N}\mu_n, \alpha)$ for all $\alpha \in I$.

If $S(\mu_1,\mu_2) = \mu_1 \vee \mu_2$, then a T-fuzzy measurable space (X,σ) is S-continuous if and only if it is continuous.

Theorem 7. *Let (X,σ) be a S-continuous T-fuzzy measurable space and m a T-fuzzy measure on (X,σ). Then for any increasing sequence $(\mu_n)_{n\in N}$*

in σ.

$$\lim_{n\to\infty}\int \mu_n dm = \int S_{n\in N}\mu_n dm.$$

Theorem 8. *Let (X,σ) be a S-continuous T-fuzzy measurable space and m a T-fuzzy measure on (X,σ). Then for any sequence $(\mu_n)_{n\in N}$ in σ,*

$$\int S_{n\in N}T_{k\geq n}\mu_k dm \leq \lim_{n\to\infty}\inf \int \mu_n dm.$$

The proofs of Theorem 7 and Theorem 8 are obtained in similar way as in the proofs of Theorem 4 and Theorem 5 by using of Definition 6.

If $S(\mu_1,\mu_2)=\mu_1\vee\mu_2$, then Theorem 4 and Theorem 5 follow immediately from Theorem 7 and Theorem 8, respectively.

In the above Proposition 2, Theorem 4, Theorem 5, Theorem 7 and Theorem 8, same results hold in case of fuzzy-valued T-fuzzy measure $\tilde{m}$.

References

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