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UNIVERSAL DERIVATION MODULE OVER QUOTIENT ALGEBRA

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Dedicated to Professor Younki Chae on his 60th birthday

1. Introduction

Throughtout the following, R will denote a commutative ring with identity 1 and A be unitary commutative R-algebra with unit and *R-derivation of A* is a mapping $D : A \longrightarrow A$ satisfying (i) $D(ra+sb) = rD(a)+sD(b)$ (R-linearity) and (ii) $D(ab) = D(a)b+aD(b)$ (multiplicative law) for all $r, s \in R$ and $a, b \in A$.

Let M be an A-module, we consider a mapping $d : A \longrightarrow_A M$ satisfying
(i) $d(ra+sb) = rd(a) + sd(b)$ and
(ii) $d(ab) = ad(b) + bd(a)$ for all $r, s \in R$ and $a, b \in A$

A-module M with such mapping d is called *A-derivation module* and will be denoted by (M, d).

Let (M_1, d_1) and (M_2, d_2) be two A-derivation modules and if there exists an A-module homomorphism $f : M_1 \longrightarrow M_2$ such that $f \cdot d_1 = d_2$ we call such f as A-derivation module homomorphism and will be denoted $f : (M_1, d_1) \longrightarrow (M_2, d_2)$. If such homomorphism is one to one and onto we call it A-derivation module isomorphism ([1,2]).

In the category of all collection of A-derivation modules and A-derivation module homomorphisms, there exists a universal elements, we call it a universal A-derivation module, explicitly for any A-derivation module (M, δ), there exists unique A-derivation module (U, d) and unique A-derivation

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module homomorphism $f : (U, d) \longrightarrow (M, \delta)$ such that $fd = \delta$ we call such (U, d) as a *universal A-derivation module.*

We denote A-Mod as the category of all left A-modules.

At first, we consider $Q_\tau(A)$ ring of quotients of A w.r.to given torsion theory τ. Then $Q_\tau(A)$ is a left A-module, thus we can regard $Q_\tau(A)$ as left R-module.

In this paper using the method developed by Golan [5], we extend A-derivation module (M, d) to $Q_\tau(A)$-derivation module $(Q_\tau(M), \bar{d})$ under a certain condition (Proposition 4). And we show that if (U, d) is universal A-derivation module, then $(Q_\tau(U), \bar{d})$ is also universal $Q_\tau(A)$ derivation module among a full subcategory of $Q_\tau(A)$-Mod. (Proposition 9). We try to see the concrete structure of universal $Q_\tau(A)$-derivation module in special case. (Proposition 10).

2. Preliminaries

Notaition and terminology concerning (hereditary) torsion theories on A-Mod will follow [4]. In particular, if τ is a torsion theory on A-Mod then a left ideal H of A is said to be τ-dense in A if and only if the cyclic left A-module R/H is τ-torsion. If M is a left A-module then we denote by $T_\tau(M)$ the unique largest submodule of M which is τ-torsion. If $E(M)$ is the injective hull of a left A-module M then we define the submodule $E_\tau(M)$ of $E(M)$ by $E_\tau(M)/M = T_\tau(E(M)/M)$. The module of quotients of M with respect to τ, denoted by $Q_\tau(M)$, is then defined to be $E_\tau(M/T_\tau(M))$. Note that, in particular, if M is τ-torsionfree then $Q_\tau(M) = E_\tau(M)$, and this is a left A-module containing M as a largest submodule. In general, we have a canonical A-homomorphism from M to $Q_\tau(M)$ obtained by composing the canonical surjection from M to $M/T_\tau(M)$ with the inclusion map into $Q_\tau(M)$.

If A is the endomorphism ring of the left A-module $Q_\tau({}_AA)$ then $Q_\tau(M)$ is canonically a left A-module for every A-module M and the canonical map $A \longrightarrow A_\tau$ is a ring homomorphism, the ring A_τ is called as the ring of quotients or localization of A at τ. A torsion theory on A-Mod is said to be faithful if and only if A, considered as a left module over itself, is τ-torsionfree. In this case, A is canonically subring of A_τ.

Lemma 1([4]). *Let H be a τ-dense ideal in A, and let $\alpha_{H,q}$ be A-module homomorphism defined on H into $Q_\tau(M)$, then R/H is τ-torsion and there exist unique R-module homomorphism $\beta_{R,q} : A \longrightarrow Q_\tau(M)$ which* *makes the daigram*

$$\begin{array}{ccccc} 0 & \longrightarrow & H & \longrightarrow & A \\ & & \alpha_{H,q} \downarrow & \swarrow \beta_{A,q} & \\ & & Q_\tau(M) & & \end{array}$$

commutes.

Lemma 2([4]). *Let H and K be τ-dense ideals of A then we have the following results.*

(1) *$H \cap K$ is τ-dense ideal.*
(2) *$(H : a) = \{r \in A \mid ra \in H\}$ is τ-dense ideal.*
(3) *Homomorphic image of H is τ-dense ideal.*
(4) *HK is τ-dense ideal.*

Lemma 3([4]). *Let H and K be τ-dense ideals of R and let $\alpha_{H,q} : H \longrightarrow Q_\tau(M)$ and $\alpha_{K,q} : K \longrightarrow Q_\tau(M)$ be defined as in the Lemma 1. Then $\alpha_{H,q}$ and $\alpha_{K,q}$ define the same element in $Q_\tau(M)$.*

3. Extension theorems

In this section we consider extensions of A-derivation module M to $Q_\tau(A)$-derivation module, in the case M is absolutely pure τ -module, where τ is a torsion theory on A-Mod.

For the given torsion theory τ on A-Mod, if a left A-module M is τ-torsionfree and τ-injective we say that M is absolutely τ-pure. We denote the class of all absolutely τ-pure A-modules by ε_τ, and we know that ε_τ is equivalent to the full subcategory of $Q_\tau(A)$-Mod (or A_τ-Mod) consisting of modules of the form $Q_\tau(M)$. Also we note that every element of ε_τ has the structure of left A_τ-module which narually extends its structure as a left A-module ([4], Proposition 6.6)

Proposition 4. *Let (M, d) be an A-derivation module and τ be a torsion theory on A-Mod and M be absolutely τ-pure left A-module, then there exists a derivation $\bar{d} : Q_\tau(A) \longrightarrow Q_\tau(M)$, the restriction of which to A is d, i.e., $(Q_\tau(M), \bar{d})$ is a $Q_\tau(A)$-derivation module.*

Proof. At first we note that $Q_\tau(M)$ is a $Q_\tau(A)$-module. If q is an element of $Q_\tau(A)$, then there exists a τ-dense ideal H of A satisfying $Hq \leq A$.

Define a function $\alpha_{H,q} : H \longrightarrow Q_\tau(M)$ by setting $\alpha_{H,q}(h) = \alpha(hq) - qd(h)$ for all h in H. Since M is absolutely τ-pure this mapping is well-defined.

We can see that $\alpha_{H,q}$ is A-homomorphism apply Lemma 1, we have that $\alpha_{H,q}$ extends uniquely to A-homomorphism from A to $Q_\tau(M)$ and so there exists unique element $\bar{q}$ of $Q_\tau(M)$ satisfying the condition $\alpha_{H,q}(h) = h\bar{q}$ for all h in H. We now define a function $\bar{d} : Q_\tau(A) \longrightarrow Q_\tau(M)$ by setting $\bar{d}(q) = \bar{q}$. This function is well-defined. Indeed, suppose that q is an element of $Q_\tau(A)$ and let H and K be τ-dense ideals of A satisfying $Hq \leq A$ and $Kq \leq A$. Then $(H \cap K)q \leq A$ and $H \cap K$ is also τ-dense ideal in A. By Lemma 3, $\alpha_{H,q}$ and $\alpha_{K,q}$ define the same element $\bar{q}$.

We have to show that such $\bar{d}$ is an R-derivation on $Q_\tau(A)$, i.e., $(Q_\tau(M), \bar{d})$ is a $Q_\tau(A)$-derivation module. Indeed, for any elements p and q in $Q_\tau(A)$ and r in R, there exist τ-dense ideals H and J of A satisfying $Hp \leq A$ and $Jq \leq A$. Take $K = H \cap J$, which is τ-dense ideal of A satisfying $Kp \leq A$ and $Kq \leq A$. For every element k of K we have

$$\begin{aligned}
\alpha_{K,p+q}(k) &= d(k(p+q)) - (p+q)d(k) \\
&= d(kp + kq) - (p+q)d(k) \\
&= \alpha_{K,p}(k) + \alpha_{K,q}(k) \\
&= (\alpha_{K,p} + \alpha_{K,q})(k)
\end{aligned}$$

By Lemma 1, the uniqueness of extension, we have that $\bar{d}(p+q) = \bar{d}(p) + \bar{d}(q)$.

Note that left A-module $Q_\tau(A)$ can be regarded as left R-module via ring homomorphism $\varphi : R \longrightarrow A$ defined by $\varphi(r) = r \cdot e$. For any element q of $Q_\tau(A)$ $r \cdot q = \varphi(r)q$.

Similarly there exists a τ-dense left ideal H of A satisfying $Hp \leq A$ and $H\varphi(r)p \leq A$. Take $K = H \cap (H : \varphi(r))$ which is τ-dense ideal of A also. Consider an A-homomorphism from K to $Q_\tau(M)$ given by $k \longrightarrow \alpha_{K,r\cdot p}(k)$

$$\begin{aligned}
\alpha_{K,r\cdot p}(k) &= d(k(r \cdot p)) - (r \cdot p)d(k) \\
&= d(k(\varphi(r)p)) - (\varphi(r)p)d(k) \\
&= d(\varphi(r)kp) - \varphi(r)(pd(k)) \\
&= \varphi(r)d(kp) - \varphi(r)pd(k) \\
&= \varphi(r)\alpha_{K,p}(k) \\
&= r \cdot \alpha_{K,p}(k)
\end{aligned}$$

Again by the uniqueness of extension, we have that $\bar{d}(rp) = r\bar{d}(p)$.

Finally take $K = HJ$, which is τ-dense ideal of A, by Lemma 2. Consider

$$\begin{aligned}\alpha_{K,pq}(kk') - p\alpha_{K,q}(kk') &= (k'q)d(kp) - pk'qd(k)\\ &= q(k'd(kp) - pk'd(k))\\ &= q((d(kk')p) - pd(kk'))\\ &= q\alpha_{K,p}(kk')\end{aligned}$$

By the uniqueness extension, we have

$$\bar{d}(pq) = p\bar{d}(q) + q\bar{d}(p).$$

Now we prove that $\bar{d}$ restricts to d on A. For any element a of A then we can take A as τ-dense ideal of A such that $Aa \leq A$. For any element b in A, which is considered τ-dense ideal of A, consider the following $\alpha_{A,a}(b) = bd(a)$. Now by the uniqueness extension, $b\bar{d}(a) = bd(a)$ so $\bar{d}(a) = d(a)$ for any element a of A.

Corollary 5. *Let τ be a torsion theory on A-Mod and (M, d) be an A-derivation module. Let M be a τ-torsionfree left A-module which is a homomorphic image of a direct sum of copies of $Q_\tau(A)$, then $(Q_\tau(M), \bar{d})$ is a $Q_\tau(A)$-derivation module, the restriction of $\bar{d}$ on A is d.*

Proof. From [5, E 26.20], M can be a left A_τ-module. Thus the mapping $\alpha_{H,q} : H \longrightarrow Q_\tau(M)$ defined in the proof of Proposition 4 is well-defined.

The remaining part follows the above proof.

From the definition of R-derivation D on A, we can consider (A, D) as an A-derivation module. The following proposition shows that (A, D) can be extended to $(Q_\tau(A), \bar{D})$, relatively weaker condition than Proposition 4.

Proposition 7. *Let τ be a faithful torsion theory on A-mod and let $D : A \longrightarrow A$ be an R-derivation, then there exists unique extension $\bar{D} : A_\tau \longrightarrow A_\tau$, which restricts to A is D.*

Proof. For any element q in $Q_\tau(A)$, there exists a τ-dense ideal H of A such that $Hq \leq A$. Now define $\alpha_{H,q} : H \longrightarrow Q_\tau(A), \alpha_{H,q}(h) = d(hq) - d(h)q$, this function is well defined.

The existence of $\bar{D}$ on A_τ follows from Proposition 4 and the fact $Q_\tau(A)$ and A_τ are isomorphic, as left A-modules. To show the uniqueness, assume that d^* and f^* be derivations defined on A_τ, and $d^* = f^*$ on A. For any non-zero element q in A_τ, there is τ-dense ideal H of A satisfying $Hq \leq A$, then for any element h in H we have $(d^* - f^*)(hq) = 0$.

From this we have that $H(d^* - f^*)(q) = 0$. Since H is τ-dense ideal of A, this implies that $d^*(q) = f^*(q)$ for all q in A_τ.

Corollary 8. *Let τ be a torsion theory on A-Mod satisfying $D(T_\tau(A)) \subseteq T_\tau(A)$. Then there exists a derivation ' $\bar{D}$ on $Q_\tau(A)$ in such a manner that the diagram*

$$\begin{array}{ccc} A & \longrightarrow & Q_\tau(A) \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle \bar{D}} \\ A & \longrightarrow & Q_\tau(A) \end{array}$$

commutes.

Proof. Define $D' : \frac{A}{T_\tau(A)} \longrightarrow \frac{A}{T_\tau(A)}$ by setting $D'(a+T_\tau(A)) = D(a)+T_\tau(A)$. By the condition $D(T_\tau(A)) \subseteq T_\tau(A)$ such a map is well-defined. And we know that $\frac{A}{T_\tau(A)}$ is τ-torsionfree, by proposition 6 there exists an extension $\bar{D}' : Q_\tau(\frac{A}{T_\tau(A)}) \longrightarrow Q_\tau(\frac{A}{T_\tau(A)})$.

Since $Q_\tau(\frac{A}{T_\tau(A)})$ is isomorphic to $Q_\tau(A)$, we have a derivation $\bar{D}$ on $Q_\tau(A)$ which making the diagram commutes.

Remark. If we take the ring R as the integer ring Z, then the R-derivation D defined on the R-algebra A can be the derivation defined in [5].

So we can say that Golan extended Z-derivation D on the noncommutative ring A to Z-derivation $\bar{D}$ on the quotient ring $Q_\tau(A)$, where A is τ-torsionfree as left A-module. ([5])

From this fact we can see some what difference between Golan's extension Theorem and ours.

4. Universal Derivation Module

Now we want to extend universal derivation module in certain subcategory of A-Mod to a full subcategory of quotient module category.

Proposition 9. *Let (U, d) be an universal A-derivation module on ε_τ, then $(Q_\tau(U), \bar{d})$ is an universal $Q_\tau(A)$-derivation module on C_τ, which is a full subcategory of $Q_\tau(A)$-Mod consisting of elements of the form M_τ.*

Proof. First we note that for any element K of C_τ, K is isomorphic to

$Q_\tau(M)$ for some left A-module M of ε_τ. For any $Q_\tau(A)$-derivation module (K, δ^*), consider that $K \cong Q_\tau(M)$ for some $M \in \varepsilon_\tau$ and $\delta^* |_A = \delta$. By the fact that (U, d) is an universal A-derivation module, there exists unique A-homorphism $f : (U, d) \longrightarrow (M, \delta)$. Since M is an element of ε_τ, we can extend f to $\bar{f} : Q_\tau(U) \longrightarrow Q_\tau(M)$ uniquely such that $\bar{f} |_U = f$ (We can prove just the same method of Proposition 4). Thus $(Q_\tau(U), \bar{d})$ is an universal $Q_\tau(A)$-derivation module.

If U is finitely generated projective A-module, then universal A-derivation module (U, d) is isomorphic to $(\mathcal{D}(A)^*, \delta)$, where $\mathcal{D}(A)^* = Hom_A(\mathcal{D}(A), A)$ and $\mathcal{D}(A) = \{D : A \longrightarrow A \mid D \textit{ is all } R - \textit{derivation on } A\}$ and A-derivation $\delta : A \longrightarrow \mathcal{D}(A)^*$ is defined by $(d(a))(D) = D(a)$ for all D in $\mathcal{D}(A)$ and a in A.

Note that if U is finitely generated projective element in ε_τ, then $Q_\tau(U)$ is also finitely generated projective $Q_\tau(A)$-module (by [4], Proposition 6.7) and $(Q_\tau(U), \bar{d})$ is also an universal derivation module among C_τ. By the same reasoning we have that $(Q_\tau(U), \bar{d})$ is isomorphic to $(\mathcal{D}(Q_\tau(A))^*, d^*)$ where $\mathcal{D}(Q_\tau(A))^* = Hom_{Q_\tau(A)}(\mathcal{D}(Q_\tau(A)), Q_\tau(A))$ and $\mathcal{D}(Q_\tau(A))$ is the set of all R-derivation on $Q_\tau(A)$ and $d^* : Q_\tau(A) \longrightarrow \mathcal{D}(Q_\tau(A))^*$ is defined by $(d^*(q))(\bar{D}) = \bar{D}(q)$ for all $\bar{D}$ in $\mathcal{D}(Q_\tau(A))$ and q in $Q_\tau(A)$.

On the other hand $\mathcal{D}(A)^* = Hom_A(\mathcal{D}(A), A)$ is isomorphic to U, then $\mathcal{D}(A)^*$ is an absolutely τ-pure A-module. By proposition 4 $(\mathcal{D}(A)^*, \delta)$ has unique extension as follows; $(Q_\tau(\mathcal{D}(A)^*), \bar{\delta})$ which is also universal element in C_τ. Thus by the uniqueness of universal element (up to isomorphic) we have the following main result.

Proposition 10. *If U is finitely generated projective universal A-derivation module among ε_τ, then $Q_\tau(\mathcal{D}(A)^*)$ is isomorphic to $\mathcal{D}(Q_\tau(A))^*$ as $Q_\tau(A)$-modules. i.e.,*

$$\begin{aligned} &Hom_{Q_\tau(A)}(\mathcal{D}(Q_\tau(A)), Q_\tau(A)) \\ &\cong Q_\tau(Hom_A(\mathcal{D}(A), A) \end{aligned}$$

as $Q_\tau(A)$-modules.

For the example satisfying the hypothesis of Proposition 10, we can take the class of τ-torsionfree finitely generated quasi-Frobenius algebras.

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