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PRIME AND SEMIPRIME IDEALS IN SEMIGROUPS

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Dedicated to Professor Younki Chae on his sixtieth birthday

1. Introduction

Pime ideals play very important role in semigroups and are rooted from prime numbers of the integers. Especially, it is cornerstone on commutative rings and topological semigroups. There is a very useful theorem known as Birkoff's subdirectly irreducible rings. In [6], the authors have proved that each semiprime ideal of a semigroup is an intersection of prime ideals. From this property, we can obtain easily Birkoff's theorem for a semigroup. Since every ring can be considered as a semigroup under multiplication, we have more generalized Birkoff's theorem. But in [6], the authors used the notion of a prime ideal in the sense of a completely prime ideal of [7]. In this paper, we want to study the relation between semiprime ideals and prime ideals in a (non-commutative) semigroup S in the sense of [7].

2. Preliminalies

S is a non-commutative semigroup and an ideal denotes always a two-sided ideal of S.

Definition 2.1. A non-empty ideal Q of a semigroup S is said to be prime if $AB \subset Q$ implies that $A \subset Q$ or $B \subset Q$ for any ideals A, B of S.

Remark. There is an analogous definitions: An ideal Q is completely prime if $ab \in Q$ implies that $a \in Q$ or $b \in Q$ for any elements a, b of S. We can

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prove easily that every completely prime ideal is prime but the converse need not be true. The primeness and completely primeness coincide if S is commutative. Someone use the notion of prime ideal in the sense of a completely prime ideal. But in this paper, we consider prime ideals in the sense of definition 2.1.

Lemma 2.2. *The following conditions are equivalent:*

1) *Q is a prime ideal of S.*
2) *$aSb \subset Q$ implies $a \in Q$ or $b \in Q$ for any a, b of S.*

Proof. 1) implies 2). Let $aSb \subset Q$. Then $(SaS)(SbS) \subset Q$ and since SaS and SbS are ideals of S, we have that $SaS \subset Q$ or $SbS \subset Q$. Assume that $SaS \subset Q$. The set $S^1aS^1 = SaS \cup \{a\}$ is the smallest ideal containing a. We can prove easily that $(S^1aS^1)^3 = (SaS \cup \{a\})^3 = (SaS \cup \{a\})(SaS \cup \{a\})\ (SaS \cup \{a\}) \subset SaS \subset Q$. So that $S^1aS^1 \subset Q$ and so $a \in Q$. Similary, if $SbS \subset Q$, we have that $b \in Q$.

2) implies 1). Let A, B be ideals of S and $AB \subset Q$. If $A \not\subset Q$, then there exists an element a in A which is not in Q. For any $b \in B$, $aSb \subset AB \subset Q$. Since $a \notin Q$, we have $b \in Q$ by 2). Thus $B \subset Q$.

The semigroup S itself is always a prime ideal of S. But S need not have proper prime ideals.

Example 2.3. Let $S = \{0, a^1, a^2, \cdots, a^m\}$, $m \geq 1$, be a semigroup with
$$a^ia^j = \begin{cases} a^{i+j} & \text{if } i+j \leq m \\ 0 & \text{if } i+j > m \end{cases}$$

Any proper ideal of S is of the form $\{a^k, a^{k+1}, \cdots, a^m, 0\} (2 \leq k \leq m)$ and is not a prime ideal of S.

Definition 2.4. A non-empty ideal P of S is said to be semiprime if $A^2 \subset P$ implies $A \subset P$ for any ideal A of S.

By the same manner to the lemma 2.2, we can prove easily that an ideal P of a semigroup S is a semiprime ideal of S if and only if $aSa \subset P$ implies $a \in P$ for any $a \in S$.

Definition 2.5. A subset T of a semigroup S is said to be an m-system if $a, b \in T$, then $axb \in T$ for some $x \in S$.

If T is a subsemigroup of S and $a, b \in T$, then $(ab)b \in TT \subset T$. Thus T is an m-system. From this fact we can consider an m-system is a generalization of a multiplicative system. The significance of this concept stems from the fact that the equivalence of lemma 2.2 asserts that an ideal

Q in a semigroup S is a prime ideal of S if and only if the compliment of Q in S is an m-system. Since S itself is a prime ideal in S, we explicitly agree that the empty set is to be considered as an m-system in order for the preceeding statement to be true without exception.

The intersection of a finite number of ideals of a semigroup is not empty. [For if A, B are ideals, we have AB is also an ideal contained in $A \cap B$]. But there are semigroups in which the intersection of all the prime ideal is empty.

Example 2.6. Let S be the set of all integers ≥ 2, the multiplication being the ordinary multiplication of numbers. The set $I(p) = \{p, 2p, 3p, \cdots\}(p =$ prime) are prime ideals of S and clearly the intersection $\cap\{I(p) | p$ runs through all primes $\}$ is empty.

We can suppose that the intersection of prime ideals is also prime if it is not empty. But it is not true. The following lemma show that it becomes a semiprime ideal of S.

Lemma 2.7. *Let Q_i be any sets of prime ideals of a semigroup $S (i \in I)$. If $P = \cap\{Q_i | i \in I\}$ is not empty, then P is a semiprime ideal of S.*

Proof. Let A be an ideal of S and $A^2 \subset P$. Then for any $i \in I$, $A^2 \subset Q_i$. Since every prime ideal is semiprime [Lemma 2.2], $A \subset Q_i$ for any $i \in I$. Hence $A \subset P$. So P is a semiprime ideal of S.

3. Main theorems

Theorem 3.1. *Every semiprime ideal of a semigroup S is an intersection of some prime ideals.*

Proof. Let P be a semiprime ideal of S and $\{Q_i | i \in I\}$ be the set of all prime ideals of S containing P. Then this set is not empty because S itself is a prime ideal of S. Let $a \notin P$, choose elements $a_1, a_2, \cdots$ inductively as follows: $a_1 = a$. Since $aSa = a_1Sa_1 \not\subset P$, take a_2 in S such that $a_2 \in a_1Sa_1$, $a_2 \notin P$. From $a_2Sa_2 \not\subset P$, we have a_3 such that $a_3 \in a_2Sa_2, a_3 \notin P, \cdots, a_{i+1} \in a_iSa_i, a_{i+1} \notin P, \cdots$. Let $A = \{a_1, a_2, \cdots\}$. Suppose that $a_i, a_j \in A$ and for convenience, let us assume that $i \leq j$. Then $a_{j+1} \in a_iSa_j$, and $a_{j+1} \in A$. A similar argument takes care of the case in which $i > j$, so we have that A is indeed an m-system and $A \cap P = \emptyset$. Now consider the set of all m-systems M of S such that $a \in M$ and $M \cap P = \emptyset$. Let $T = \{M | M$ is an m-system of S and $a \in M$, $M \cap P = \emptyset\}$. Then this set T is non-empty. By Zorn's Lemma there exists a maximal element, say M' in T. Again let $X = \{J \mid J$ is an ideal of S and $J \cap M' = \emptyset$, $J \subset P\}$. Then X is non-empty since P is in X. If we use Zorn's lemma once more on the set X, there exists an maximal element, say Q in X. If $x, y \in S \backslash Q$ then $(S^1xS^1 \cup Q) \cap M' \neq \emptyset$, $(S^1yS^1 \cup Q) \cap M' \neq \emptyset$ since $S^1xS^1 \cup Q$ and $S^1yS^1 \cup Q$ are ideals of S properly containning Q. Hence there are some elements s, t, u, v in S' such that $sxt, uyv \in M'$. Since M' is an m-system, there is an element m in S such that $sxtmuyv \in M'$. From the fact $sxtmuyv \notin Q$, $xtmuy \notin Q$. Hence $xtmuy \in (S \backslash Q)$ and so we have that $S \backslash Q$ is an m-system. From the maximality of M', $S \backslash Q = M'$ and so Q is a prime ideal of S containing P. Since $a \notin Q$, this means $P \supset \cap\{Q_i \mid i \in I\}$. Since the converse inclusion is trivial, we have that $P = \cap_{i \in I}\{Q_i | Q_i$ is a prime ideal of S containing $P\}$. In other word, a semiprime ideal P of S is the intersection of all prime ideals of S containing P.

Definition 3.2. A non-empty ideal I of S is said to be completely semiprime if $a^2 \in I$ implies $a \in I$ for any $a \in S$.

Corollary 3.3. *Any completely semiprime ideal of S is an intersection of prime ideals of S.*

The following result is fairly similar to the proof of Theorem 3.1. But for the sake of complete, we write out a proof.

Theorem 3.4. *Any completely semiprime ideal of S is an intersection of completely prime ideals of S.*

Proof. Let I be an any completely semiprime ideal of S, $a \notin I$ and $A = \{a, a^2, \cdots\}$. Then A is an m-system and $A \cap I = \emptyset$. Using Zorn's lemma for the set of all m-system which contains a and has no intersection with I. We have a maximal m-system M'. By the same method given in Theorem 1, if we put Q as a maximal ideal of S containing I which has no intersection with M', we have that the compliment of Q is M'. Let $< M' >$ be a subsemigroup of S generated by M'. If $< M' > \cap I \neq \emptyset$, then there are $a_1, a_2, \cdots, a_n \in M'$ such that $a_1a_2 \cdots a_n \in I$. Since M' is an m-system, there are $x_1, x_2, \cdots, x_{n-1} \in S$ such that $a_1x_1a_2x_2 \cdots a_{n-1}x_{n-1}a_n \in M'$. Since I is a completely semiprime ideal, $ab \in I$ implies $ba \in I$. So that we have $a_1x_1a_2x_2 \cdots a_{n-1}x_{n-1}a_n \in I$. But this is contradiction. Hence M' is a subsemigroup of S. So that Q is a completely prime ideal of S and this complete the proof.

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