Kyungpook Mathematical Journal
Volume 32, Number 3, October, 1992.

AN ISOPARAMETRIC SUBMANIFOLD AND ITS TOPOLOGICAL STRUCTURE

Kwang Sung Park

Dedicated to Professor Younki Chae on his 60th birthday

1. Introduction

Isoparametric families in constant curvature spaces are given as level sets of isoparametric functions. Also each regular level set, an isoparametric hypersurface, is also characterized by having constant principal curvatures. The study of isoparametric hypersurfaces was initiated by E. Cartan, and plays an important role in submanifold theory. In early 70's, H. Münzner showed that the only possible numbers of distinct principal curvatures are 1, 2, 3, 4 and 6. In [5], Park studied transnormal systems on projective spaces. The aim of this paper is to improve some of his proofs and results on CP^3. We are, in particular, interested in a transnormal system on CP^3 which is the image of $\overline{M}$ under the Hopf fibration $S^7 \longrightarrow CP^3$, where $\overline{M}$ is an isoparametric hypersurface of S^7 with 6 distinct principal curvatures. We proved that any non-singular foil of this transnormal system has 5 nonconstant distinct principal curvatures which improve the results in [5]. And we used much simpler methods to prove these results.

2. Preliminaries

A *transnormal system* on a complete connected Riemannian manifold N is a partition of N into nonempty connected submanifolds, called *foils*,

Received March 30, 1992.

This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1990.

such that any geodesic of N cuts the foils orthogonally at none or all of its points. Consider the unit sphere S^{2m+1} as a subspace of $R^{2m+2} = C^{m+1}$. Let $\overline{M}$ be a hypersurface of S^{2m+1} and $\pi : S^{2m+1} \longrightarrow CP^m$ the Hopf fibration. $\overline{M}$ is said to be *isoparametric* if all of its principal curvatures are constant. For each tangent vector $\overline{X}$ on S^{2m+1}, $v\overline{X}$ and $h\overline{X}$ denote the vertical and horizontal components of $\overline{X}$, respectively. To each tengent vector field X on CP^m, there exists a unique horizontal vector field $\overline{X}$ on S^{2m+1} such that $(\pi_*)_z\overline{X}_z = X_{\pi(z)}$ for all $z \in S^{2m+1}$.

Let $\overline{D}$ and D be the Riemannian connections of S^{2m+1} and CP^m, respectively. If X and Y are tangent vector fields on CP^m, then we have $h(\overline{D}_{\overline{X}}\overline{Y}) = \overline{D_XY}$, where $\overline{X}, \overline{Y}, \overline{D_XY}$ mean their horizontal lifts.

Let M be a hypersurface of CP^m, and $\tilde{\nu}$ a given unit normal vector field on $\overline{M} = \pi^{-1}(M)$. Let $\nu = \pi_*(\tilde{\nu})$, then the relationship between the two shape operators $A_{\tilde{\nu}}$ and A_ν is given by $h(A_{\tilde{\nu}}\overline{X}) = \overline{A_\nu X}$, where X is a tangent vector on M.

Let $\mathcal{L}$ be a transnormal system on CP^m which contains a foil M of codimension 1, and $\overline{M} = \pi^{-1}(M)$. Then $\pi^{-1}(\mathcal{L})$ is a system which contains a foil $\overline{M}$ of codimension 1. In fact, we have the following

Proposition 1([4], [7]). *$\pi^{-1}(\mathcal{L})$ is a transnormal system on S^{2m+1}. And hence it is an isoparametric family.*

Thus $\overline{M}$ is an isoparametric hypersurface of S^{2m+1}. According to [3], there exist exactly two focal submanifolds $\overline{F_\pm}$ with codimensions $m_\mp$.

3. Results

Suppose that $\overline{M}$ has 6 distinct principal curvatures. Then $m = 6k$ and $m_+ = m_- = k$, where $k = 1, 2$. Let $F_+ = \pi(\overline{F}_+)$. If $m_\pm = 2$, then we obtain easily that $H^q(\overline{F}_+; Z_2) = 0$ for odd q ([3]). But the Gysin exact sequence from the fibration $S^1 \longrightarrow \overline{F}_+ \longrightarrow F_+$ gives a contradiction ([5]). Thus we have the following

Proposition 2 [5]. *Let M be a hypersurface in a transnormal system on CP^m such that $\pi^{-1}(M)$ is a hypersurface in sphere with 6 distinct principal curvatures, then $m_\pm = 1$.*

From now on we restrict our attention to the case that $\overline{M}$ has 6 distinct principal curvatures $p_1, \cdots, p_6$. Let $T(p_i)$ denote the eigenspace corresponding to the eigenvalue p_i. Let V be the canonical vertical vector field on $\overline{M}$. Let k be the number of eigenspaces $T(p_i)$ containing a non-horizontal vector. According to [5], the number k plays an important role to study transnormal systems on projective spaces. Also he proved that $k \geq 2$. Therefore the possible values of k are 2, 3, 4, 5 or 6. We may assume that $p_1 > p_2 > p_3 > p_4 > p_5 > p_6$ and let $p_1 = \cot t$, then $p_i = \cot(t + (i-1)\pi/6)$. And the shape operator A is represented by the matrix

$$\begin{pmatrix} a_1 + p_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 + p_2 & \cdots & a_2 \\ & \cdots & \cdots & \\ a_5 & a_5 & \cdots & a_5 + p_5 \end{pmatrix}$$

Now we examine each case of the possible values of k.

(1) $k = 2$.

Then $J\nu$ is principal. Clearly $T(p_i)$ and $T(p_{i+3})$ are non-horizontal spaces for some i. But we may assume that $T(p_2)$ and $T(p_5)$ are the two non-horizontal spaces. Since $p_2 + p_5 = p_3$ for some t, by the following lemma, a focal submanifold of M must have dimension 3. It's a contradiction.

Lemma 3[1]. *Let M be a hypersurface in CP^m. Suppose that $J\nu$ is an eigenvector of A_ν with the eigenvalue $2\cot 2t$. Let ϕ_t be the normal exponential map on M. If $X = J\nu$ or $A_\nu X = (\cot t)X$, then $(\phi_t)_* X = 0$.*

(2) $k = 3$.

After a routine computation, we obtain that $T(p_i), T(p_{i+2}), T(p_{i+4})$ are the non-horizontal spaces, where $i = 1$ or 2. We may assume that $i = 1$. Let $t = \pi/12$, then the characteristic polynomial of A is given by $f(x)(x^2 - 3x - 2)$, where $f(x)$ is a suitable product of linear factors. The following identities are known in [5].

$$a_1 p_1 + a_3 p_3 = -1 \quad \textit{and} \quad a_1 + a_3 = p_5 = -1$$

By using horizontal parts in vectors in $T(p_3), T(p_5)$, we can compute a_1. This leads to a contradiction to the above identities.

(3) $k = 4$.

There are essentially three possible cases. If $T(p_5), T(p_6)$ are the two horizontal eigenspaces, then we have

$$a_2(p_2 - p_1) + a_3(p_3 - p_1) = -(1 + p_1 p_4) = 0.$$

But we can easily get $a_2, a_3 < 0$. It's a contradiction. The other cases, that is, $T(p_4)$ or $T(p_3)$ instead of $T(p_5)$ is horizontal, can be examined by a similar argument. Thus $k = 4$ is impossible.

(4) $k = 5$.

In this case, we get the following identities.

$$a_1 + a_2 + a_3 + a_4 = p_5$$

$$a_1 p_1 + a_2 p_2 + a_3 p_3 + a_4 p_4 = -1$$

$$a_1(p_1 - p_2) + a_3(p_3 - p_2) + a_4(p_4 - p_2) = 0$$

Also we have $a_i(K_i + 1) = p_4 - p_1 < 0$ for some positive number K_i. Thus $a_1, \cdots, a_4 < 0$. From the last identity of the above, we get a contradition.

Therefore the only possible case is $k = 6$. Actually this is possible (see [5]). Now we assume that $k = 6$, that is, there are no horizontal eigenspaces. Then we get easily the following

Proposition 4 [5]. *Let M be hypersurface in a transnormal system on CP^3 such that $\pi^{-1}(M)$ has 6 distinct principal curvatures. Then $\pi^{-1}(M)$ has no horizontal eigenspaces and M has 5 principal curvatures and at least one of them is nonconstant.*

Now we are going to prove that these 5 principal curvatures are nonconstant. Let X be an eigenvector with a principal curvature p and $\overline{X}$ its horizontal lift. If p is constant, then we have

$$\overline{X} = b_1 U_1 + b_2 U_2 + b_3 U_3 + b_4 U_4 + b_5 U_5,$$

where b_i are constant. Since $\overline{A}(\overline{X}) = p\overline{X} + \textit{vertical part}$, we get

$$p_i + a_i \sum_{j=1}^{5} b_j = \textit{constant}$$

from the representation matrix of the shape operator. Then all a_i's are constant, and hence all principal curvatures of M is constant. It is a contradition. Thus we have the following

Proposition 5. *Let M be as in Proposition 4. Then every principal curvature of M is nonconstant.*

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DEPARTMENT OF MATHEMATICS, KEIMYUNG UNIVERSITY, TAEGU 704-701, KOREA.