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CONTACT MANIFOLDS WITH $\phi R = R\phi$

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Dedicated to Professor Younki Chae on his sixtieth birthday

1. Introduction

Let M^{2n+1} be a contact metric manifold and (ϕ, ξ, η, g) be its contact metric structure (see section 2 for definitions.) M^{2n+1} is said to be Sasakian if the structure is normal. On a Sasakian manifold the Ricci tensor R commutes with ϕ ([1, p.76]). The converse question, "Is a contact manifold with $R\phi = \phi R$ Sasakian ? " is still open. On the other hand, generalizing Theorem 3.3 of Okumura ([6]), Tanno ([9,10]) proved that every conformally flat K-contact manifold is a space form. The results concerning the Ricci and scalar curvatures of a conformally flat contact meric manifold have been obtained in [7] and [11]. Recently Blair and Koufogiorgos ([2]) gave a partial answer about this problem as follow.

A conformally flat contact metic manifold with $R\phi = \phi R$ is of constant curvature.

The purpose of the present paper is to prove the following Theorem corresponding to Blair and Koufogiorgos' result, replacing conformal flatness by vanishing contact conformal curvature tensor field (see section 2 for definition).

Theorem. *Every contact metric manifold $M^{2n+1}(n > 2)$ with vanishing contact conformal curvature tensor field and $R\phi = \phi R$ is of constant ϕ-sectional curvature $\{s-n(3n+1)\}/n(n+1)$, where s is the scalar curvature.*

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Furthermore, generalizing Theorem 4.2 of Blair and Koufogiorgos ([2]), we obtain the following Theorem.

Theorem. *Let M^{2n+1} be a contact metric manifold with $R\phi = \phi R$. Then the followings are equivalent to each other.*

(1) *M^{2n+1} is an Einstein manifold.*
(2) *M^{2n+1} has parallel Ricci tensor.*
(3) *M^{2n+1} has harmonic curvature.*

2. Preliminaries

Let (M, ϕ, ξ, η, g) be a $(2n+1)$-dimensional almost contact manifold, that is, M is a manifold covered by a system of coordinate neighborhood $\{u; x^h\}$ and (ϕ, ξ, η, g) an almost contact metric structure on M, formed by ϕ, ξ, η tensor of type $(1,1), (1,0)$ and $(0,1)$, respectively, and a Riemannian metric g such that

$$(2.1) \qquad \left\{ \begin{array}{lll} \phi_j^i \phi_i^h = -\delta_j^h + \eta_j \xi^h, & \phi_j^h \xi^i = 0, & \eta_i \phi_j^i = 0, \\ \eta_i \xi^i = 1, & g_{ts} \phi_j^t \phi_i^s = g_{ji} - \eta_j \eta_i, & \eta_i = g_{ih} \xi^h. \end{array} \right.$$

If an almost contact metric structure satisfies

$$(2.2) \qquad \phi_{ji} = \frac{1}{2}(\nabla_j \eta_i - \nabla_i \eta_j),$$

where ∇_j denotes the operator of covariant differentiation with respective to g_{ji}, then the almost contact metric structure is called a cotact metric structure. A manifold with a contact metric structure is called a contact metric manifold. A manifold with a normal contact metric structure is called a Sasakian manifold.

Let M^{2n+1} be a contact metric manifold. Then we can consider the contact conformal curvature tensor field C_0 on M (the same definition as the contact conformal curvature tensor field in [4]).

$$\begin{aligned}(2.3) \quad C_{0,kjih} = \; & R_{kjih} + \frac{1}{2n}(g_{kh}R_{ji} - g_{jh}R_{ki} + R_{kh}g_{ji} \\ & -R_{jh}g_{ki} - R_{kh}\eta_j\eta_i + R_{jh}\eta_k\eta_i - \eta_k\eta_h R_{ji} \\ & +\eta_j\eta_h R_{ki} - \phi_{kh}S_{ji} + \phi_{jh}S_{ki} - S_{kh}\phi_{ji} \\ & +S_{jh}\phi_{ki} + 2\phi_{kj}S_{ih} + 2S_{kj}\phi_{ih}) \\ & +\frac{1}{2n(n+1)}\{2n^2 - n - 2 + \frac{(n+2)s}{2n}\}(\phi_{kh}\phi_{ji}\end{aligned}$$

$$
\begin{aligned}
&-\phi_{ki}\phi_{jh} - 2\phi_{kj}\phi_{ih}) \\
&+\frac{1}{2n(n+1)}\{n+2-\frac{(3n+2)s}{2n}\}(g_{kh}g_{ji}-g_{ki}g_{jh}) \\
&-\frac{1}{2n(n+1)}\{4n^2+5n+2-\frac{(3n+2)s}{2n}\}(g_{kh}\eta_j\eta_i \\
&-g_{ki}\eta_j\eta_h+\eta_k\eta_h g_{ji}-\eta_k\eta_i g_{jh}),
\end{aligned}
$$

where (ϕ, ξ, η, g) denotes the contact metric structure, R_{kjih}, R_{ji} and s are Riemannian curvature tensor, Ricci tensor and scalar curvature of M, respectively, and $S_{ji} = \phi_j^t R_{ti}$.

A plane section in $T_x(M)$ is called a ϕ-section if there exist a unit vector X in $T_x(M)$ orthogonal to ξ such that $\{X, \phi X\}$ is an orthonormal basis of the plane section. Then the sectional curvature $g(R(X,\phi X)\phi X, X)$ is called a ϕ-sectional curvature.

A Rimannian curvature tensor is said to be harmonic if it the Ricci tensor R_{ji} satisfies the Codazzi equation, namely, in local coordinate, $\nabla_k R_{ji} = \nabla_j R_{ki}$, where $\nabla_k R_{ji}$ denotes the covariant derivative of Ricci tensor R_{ji}. This condition is essentially weaker than that for the parallel Ricci tensor. Recently Riemanian manifolds with harmonic curvature are studied by A. Derdziński ([3]), H. Nakagawa and U-H. Ki ([5]), E.Ômachi ([8]) and others.

3. Proof of Theorems

First, we assume that the contact metric structure satisfies $\phi R = R\phi$ and the contact conformal curvature tensor filed vanishes identically.

Then we have

$$\phi_j^t R_t^i = R_j^t \phi_t^i \qquad (\text{ i.e.,} S_{ij} = -S_{ji}). \tag{3.1}$$

Transvecting ξ^j to (3.1) and using (2.1), we have

$$R_{ji}\xi^i = \alpha\eta_j, \tag{3.2}$$

where $\alpha = g(R\xi, \xi) = R_{ji}\xi^j\xi^i$.

On the other hand, C_0 implies

$$
\begin{aligned}
0 \;=\;& R_{ji} + \frac{1}{2n}\{(2n+1)R_{ji} - R_{ji} + sg_{ji} - R_{ji} \\
(3.3) \qquad & -s\eta_j\eta_i + \alpha\eta_j\eta_i - R_{ji} + \alpha\eta_j\eta_i
\end{aligned}
$$

$$
\begin{aligned}
&+\phi_j^s\phi_s^t R_{ti} + \phi_j^t R_t^s \phi_{si} - 2\phi_{js}\phi_i^t R_t^s - 2\phi_j^t R_{ts}\phi_i^s\} \\
&+\frac{1}{2n(n+1)}\{2n^2 - n - 2 + \frac{(n+2)s}{2n}\}(3g_{ji} - 3\eta_j\eta_i) \\
&+\frac{1}{2n(n+1)}\{n + 2 - \frac{(3n+2)s}{2n}\}2ng_{ji} \\
&+\frac{1}{2n(n+1)}\{-4n^2 - 5n - 2 + \frac{(3n+2)s}{2n}\}\{(2n-1)\eta_j\eta_i + g_{ji}\},
\end{aligned}
$$

from which, taking account of

$$
\begin{aligned}
\phi_j^t R_t^s \phi_{si} &= -R_{ji} + \alpha\eta_j\eta_i, \\
-2\phi_{js}\phi_i^t R_t^s &= -2R_{ji} + 2\alpha\eta_j\eta_i, \\
-2\phi_j^t R_{ts}\phi_i^s &= -2R_{ji} + 2\alpha\eta_j\eta_i,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
0 &= 2(n-2)R_{ji} + \{2(n-2) - \frac{(n-2)s}{n}\}g_{ji} \\
&\quad + \{4\alpha + \frac{(n-2)s}{n} - 2(2n^2 + n - 2)\}\eta_j\eta_i.
\end{aligned} \tag{3.4}
$$

Transvecting with ξ^i to (3.4), we have

$$
\alpha = 2n, \qquad R_{ji}\xi^i = 2n\eta_j. \tag{3.5}
$$

Thus we have

Lemma 3.1. *On a contact metric manifold $M^{2n+1}(n > 2)$ with vanishing contact conformal curvature tensor field, if $R\phi = \phi R$, then*

$$
R_{ji} = \left(\frac{s}{2n} - 1\right)g_{ji} + \left(2n + 1 - \frac{s}{2n}\right)\eta_j\eta_i. \tag{3.6}
$$

Substituting (3.6) in $C_0 = 0$, we obtain

$$
\begin{aligned}
R_{kji}{}^h &= \frac{\lambda+3}{4}(\delta_k^h g_{ji} - \delta_j^h g_{ji}) + \frac{\lambda-1}{4}(\phi_k^h\phi_{ji} - \phi_j^h\phi_{ki} \\
&\quad -2\phi_{kj}\phi_i^h - \delta_k^h\eta_j\eta_i + \delta_j^h\eta_k\eta_i - \xi_k\xi^h g_{ji} + \eta_j\xi^h g_{ki}),
\end{aligned}
$$

where $\lambda = \frac{1}{n(n+1)}\{s - n(3n+1)\}$.

We now prove that the scalar curvature s is constant under our assumptions. Denoting by $\mathcal{L}$ Lie differentiation we define the operator h by $h = \frac{1}{2}\mathcal{L}_\xi \phi$ (cf. [1]). The (1.1) tensor h is self-adjoint and satisfies

$$(3.7) \qquad \nabla_j \xi^i = \phi_j^i + \phi_j^t h_t^i, \qquad \mathrm{Tr}\ h\phi = 0, \qquad h\phi = -\phi h.$$

Thus we have

$$(3.8) \qquad \nabla_t \xi^t = 0.$$

On the other hand, differentiating (3.5) and using (3.6), we can obtain

$$\frac{n-1}{2n}\{\nabla_i s - (\xi^t \nabla_t s)\eta_i\} = 0$$

and

$$\xi^t \nabla_t s = 2(2n + 1 - \frac{s}{2n})\nabla_t \xi^t,$$

which together with (3.8) implies $\nabla_i s = 0$, that is, s is constant.

Thus we have

Theorem 3.2. *Every contact metric manifold $M^{2n+1} (n > 2)$ with vanishing contact conformal curvature tensor field and $R\phi = \phi R$ is of constant ϕ-sectional curvature $\{s - n(3n+1)\}/n(n+1)$, where s is the scalar curvature.*

Finally, we assume that M^{2n+1} is a contact metric manifold with $R\phi = \phi R$.

It is well-known ([2]) that M^{2n+1} is Einstein if and only if M^{2n+1} has parallel Ricci tensor with $R\phi = \phi R$. On the other hand, it is clear that M^{2n+1} has harmonic curvature tensor if M^{2n+1} has parallel Ricci tensor.

Let M^{2n+1} be a contact metric manifold with $R\phi = \phi R$ and harmonic curvature. Then differentiating (3.2), we have

$$(3.9) \qquad (\nabla_j R_{it})\xi^t + R_{it}\nabla_j \xi^t = (\nabla_j \alpha)\eta_i + \alpha \nabla_j \eta_i.$$

Taking the skew-symmetric part with respect to the indices j and i, we have

$$(3.10) \qquad R_{it}\nabla_j \xi^t - R_{jt}\nabla_i \xi^t = (\nabla_j \alpha)\eta_i - (\nabla_i \alpha)\eta_j + 2\alpha\phi_{ji}.$$

Transvecting with ξ^i to (3.10) and using (2.2), (3.7), we have

$$(3.11) \qquad 2\alpha\phi_{ji} = 2R_{it}\phi_j^t + R_{it}\phi_j^s h_s^t - R_{jt}\phi_i^s h_s^t.$$

Transvecting with ϕ_k^j to (3.11) and using (2.1), we have

$$R_{ik} = \alpha g_{ik}. \tag{3.12}$$

Thus we have

Theorem 3.3. *Let M^{2n+1} be a contact metric manifold with $R\phi = \phi R$. Then the followings are equivalent to each other.*

(1) *M^{2n+1} is an Einstein manifold.*
(2) *M^{2n+1} has parallel Ricci tensor.*
(3) *M^{2n+1} has harmonic curvature.*

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