Kyungpook Mathematical Journal
Volume 32, Number 3, October, 1992.

ON NONLINEAR CAUCHY-KOWALEWSKI THEOREM

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Dedicated to Professor Younki Chae on his sixtieth birthday

In this paper we have shown that if $u \in C^1$, then the Cauchy problem $\frac{\partial u}{\partial t} = F(t, x, u, \nabla u), u(0, x) = 0$ has a unique C^1-solution in a certain Banach space X where F is holomorphic with respect to $x \in R^n, u, \nabla_x u$ and continuous with respect to $t \in R$. u can be either single valued or vector valued function.

Consider the following nonlinear Cauchy problem:

$$\frac{\partial u}{\partial t} = F(t, x, u, \nabla_x u) \tag{1}$$

$$u(0, x) = u_0(x) \tag{2}$$

where $x = (x_1, x_2, \cdots, x_n) \in R^n$, F is holomorphic with respect to x, u, $\nabla_x u = (u_{x_1}, u_{x_2}, \cdots, u_{x_n})$, and continuous with respect to $t \in R$. u and F are either single or vector valued functions. Notice that Baouendi-Goulaouic-Treves([1]), Metivier ([4]) deal with single valued functions u and F, while Nirenberg([7]), Ovsjannikov([10], [11]) and Treves ([12]) treat u and F as vector valued functions. In [1], [4] F is extended to a vector valued function.

Treves([12]) has shown that the Cauchy problem (1)-(2) has a unique analytic solution in the scale of Banach space provided F is analytic with

Received March 23, 1992.

respect to t, holomorphic with respect to the rest of variables and the initial data $u_0(x)$ is also analytic. The scale of Banach space was introduced by Ovsjannikov([10,11]) (see also Gelfand-Shilov ([3])). Later on Baouendi-Goulaouic-Treves ([1]) have introduced the generalized Hamilton vector field M_i and proved that any C^2-solution of (1)-(2) is analytic and therefore unique if the system (1) is in involution, i.e., $M_iF_j = M_jF_i$, $F = (F_1, F_2, \cdots, F_m)$, $i, j = 1, 2, \cdots, m$ and the initial data $u_0(x)$ is analytic. Furthermore if (1) is semilinear the above assertion holds for $u \in C^1$. Notice that the requirement $u \in C^2$ is necessary for the definition of the generalized Hamilton field M_i. They have also given an approximate C^2-solution via the Gaussian kernel when the space dimension is 1 and $u_0(x) \in C^2$. Later on Metivier ([4]) has extended the approximation formula of Baouendi-Goulauic-Treves to any space dimension $n > 1$ by introducing more generalized Hamilton fields. He then showed that if there are two C^2-solutions u and u^* of (1)-(2), then $u = u^*$. Moreover he showed the unique C^2-solution of (1)-(2) is approximated by a sequence of analytic functions via the Gaussian kernel for any space dimension $n > 1$.

Let u be holomorphic. Then Nirenberg ([7]) showed that the Cauchy problem (1)-(2) has a unique analytic solution (analytic in $x \in R^n$, continuously differentiable in $t \in R$) in the Banach space X_s under the following assumptions on F:

For some numbers $R > 0$, $T > 0$, and every pair of numbers s, s' such that $0 \le s' < s < 1$, $(u, t) \to F(u, t)$ is a continuous mapping of

$$\{u \in X_s : \|u\|_s < R\} \times \{t : |t| < T\} \text{ into } X_{s'}. \tag{3}$$

Secondly, for any positive $s < 1$ and every $u \in X_s$ with $\|u\|_s < R$ and for any t, $|t| < T$, there is a linear operator $A_u(t)$ mapping X_s into X_s, with

$$\|A_u(t)v\|_{s'} \le C\frac{\|v\|_s}{s - s'} \text{ for every } 0 \le s' < s \tag{4}$$

such that for any $\|v\| < R$,

$$\|F(v,t) - F(u,t) - A_u(t)(v - u)\| \le C\frac{\|v - u\|_s^{1+\delta}}{s - s'} \tag{5}$$

This is to hold for every $s' < s$ and with fixed positive constants $\delta \le 1$ and C independent of t, u, v, s or s'.

Finally: $F(0,t)$ is a continuous function of t, $|t| < T$ with values in X_s for every $s < 1$ and satisfying with a fixed constant K,

$$\|F(0,t)\|_s \leq \frac{K}{1-s}, \quad 0 \leq s < 1. \tag{6}$$

Remark. Nishida ([9]) has slightly improved inequality (5) as follows:

$$\|F(u,t) - F(v,t)\|_{s'} \leq C\frac{\|u-v\|_s}{s-s'}$$

where C is a constant independent of t, u, v, s, s'.

Let $u = (u_1, u_2, \cdots, u_N)$ be vector-valued holomorphic. Then (1) is easily transformed to a quasilinear equation:

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{N} a^j(t,x,u)u_{x_j} + b(t,x,u) \tag{7}$$

where a^j is an $N \times N$-matrix, u_{x_j} and $b(t,x,u)$ are $N \times 1$-column vectors. Define

$$\|u\|_s = \sup_{D_s} |u(x)|, \text{ where } D_s = \prod_{0<s<1} \{|x_j| < sR\}$$

Let X_s be the Banach space endowed with the norm defined as above. Then Nirenberg ([7]) proved that equation (7) satisfies assumptions (4)-(5) from which the existence and uniqueness of the Cauchy problem (1)-(2) follows in the Banach space X_s (see [7: Theorem 1.1]).

The natural question is: What is the minimum number of derivatives required to have existence and uniqueness of solutions of the Cauchy problem (1)-(2)? In this paper we tackle the above question. We use the following norm introduced by Ovsjannikov ([10, [11], see also [3]) for $u \in C^m (1 \leq m \leq \infty)$:

$$\|u\|_s = \sup_{|\alpha| \leq m} \frac{s^{|\alpha|}}{\alpha!}\|D^\alpha u\|_0, \quad \|D^\alpha u\|_0 = \sup_{D_s} |D^\alpha u|, s < 1. \tag{8}$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$. Let $X_s^{(m)}$ be the Banach space of $u \in C^m$ equipped with the norm defined by (8). Then $X_s^{(m)} \subset X_{s'}^{(m)}$ for $s' \leq s$ and the natural injection $X_s^{(m)} \to X_{s'}^{(m)}$ has norm ≤ 1. The differential operator ∂_x maps $X_s^{(m)}$ into $X_{s'}^{(m)}$ as follows:

Lemma 1. *Let $u \in X_s^{(m)}$ for $1 \leq m \leq \infty$. Then*

$$\|\partial_x u\|_{s'} \leq C\frac{\|u\|_s}{s-s'} \text{ for each } 0 \leq s' < s < 1 \tag{9}$$

Proof.

$$\begin{aligned}
\|\partial_x u\|_{s'} &= \sup_{1\leq|\alpha|\leq m} \frac{s'^{|\alpha|-1}}{(\alpha-1)!}\|\partial_x^{|\alpha|-1}\partial_x u\|_0 \\
&= \sup_{0\leq|\alpha|\leq m} (|\alpha|/s')(s'/s)^{|\alpha|}(s^{|\alpha|}/\alpha!)\|\partial_x^\alpha u\|_0 \\
&\leq \sup_{0\leq|\alpha|\leq m} (s^{|\alpha|}/\alpha!)\|\partial_x^\alpha u\|_0 \sup_{0\leq|\alpha|\leq m} (|\alpha|/s')(s'/s)^{|\alpha|} \\
&= \sup_{0\leq|\alpha|\leq m} (|\alpha|/s')(s'/s)^{|\alpha|}\|u\|_s \text{ by (8).} \qquad (10)
\end{aligned}$$

Now

$$\begin{aligned}
\sup_{0\leq|\alpha|} |\alpha|(s'/s)^{|\alpha|} &\leq (1/e)(1/\ln(s/s')) \\
&= (1/e)(1+\varepsilon)(s'/(s-s')), 0 \leq \xi \leq (s-s')/s' \\
&\leq C\frac{s'}{s-s'} \qquad (11)
\end{aligned}$$

Observe that $\xi \to 0$ when $s' \to s$. When $s' \to 0$, both sides trivially go to zero.

In order words the constant C in inequality (11) is a bounded constant for given s and s' with $s' < s$. Substitution of (11) into (10) completes the proof.

Theorem 1. *Suppose F satisfies the Nirenberg conditions (3)-(6). Let $u(x,\cdot)$ and $u_0(x)$ belong to $X_s^{(1)}(0 \leq s < 1)$ with $\|u(x,\cdot)\|_s$, $\|u_0(x)\|_s \leq \frac{1}{C}$ where C is the constant given by (9). Then the Cauchy problem of the quasilinear equation (7) with the initial data $u_0(x)$ has a unique solution in $X_s^{(1)}$.*

Proof. A careful analysis of Nirenberg's proof (7: pp.566-571) shows that his proof also works for $u \in X_s^{(1)}$ which asserts the existence of solutions in $X_s^{(1)}$. To claim the uniqueness of solutions we must therefore show that conditions (4) -(5) hold for $u \in X_s^{(1)}$. Since $F(u,t)$ is given by the right hand side of (7) we get

$$(12)\ F(v,t) = a^j(t,x,u)V_{x_j} + \sum_{i=1}^{N} a^j_{u_i}(t,x,u)(v_i - u_i)v_{x_j} + b(t,x,u) + \sum_{i=1}^{N} b_{u_i}(t,x,u)(v_i - u_i) + 0(|v-u|^2) + 0(|v-u|^2 v_{x_j})$$

The linear approximation of $F(v,t) - F(u,t)$ is thus majorized from (12) by

$$
\begin{aligned}
(13)\qquad & |F(v,t) - F(u,t) - a^j(t,x,u)(v_{x_j} - u_{x_j}) - \\
& \sum_{i=1}^{N}(a^j_{u_i}(t,x,u)v_{x_j} + b_{u_i}(t,x,u))(v_i - u_i)| \\
& \le C_1\{|v-u|^2 + |v-u|^2|v_x|\} \\
& \le C_1\{|v-u|^2(1+|v_x|)\} \\
& \le C_1\{\|v-u\|^2_{s'}, (1+\|v_x\|_{s'})\} \\
& \le C_1\{\|v-u\|^2_{s}, (1+\|v_x\|_{s'})\} \text{ a fortiori} \\
& \le C_1\{\|v-u\|^2_{s}, (1+C\frac{\|v\|_s}{s-s'})\} \text{ by Lemma 1} \\
& \le 2C_1\frac{\|v-u\|^2_s}{s-s'} \text{ by the assumption } \|v\|_s \le \frac{1}{C}
\end{aligned}
$$

which is the desired inequality (5) with the constant $2C_1$ and $\delta = 1$. The linear part $A_u(t)w$ according to (12) and (13) is majorized by:

$$
\begin{aligned}
\|A_u(t)w\|_{s'} &= \|a^j(t,x,u)w_{x_j} + \sum_{i=1}^{N}(a^j_{u_i}(t,s,u)u_{x_j} + b_{u_i}(t,x,u))w_i\|_{s'} \\
&\le C_2\{\|w_x\|_{s'} + (\|u_x\|_{s'} + 1)\|w\|_{s'}\} \\
&\le C_2\{C\frac{\|w\|_s}{s-s'} + (1 + C\frac{\|u\|_s}{s-s'})\|w\|_s\} \text{ by Lemma 1} \\
&\le C_2(\frac{C}{s-s'} + \frac{2}{s-s'})\|w\|_s \text{ by the assumption } \|u\| \le \frac{1}{C} \\
&= C_2(2+C)\frac{\|w\|_s}{s-s'}
\end{aligned}
$$

which is the desired inequality (4). Consequently $F(u,t)$ satisfies all the Nirenberg conditions (3)-(6). This compleste in view of Theorem 1.1 of [7] the proof.

Since we can always transform fully nonlinear equation (1) to a quasilinear equation (7) provided $u \in C^2$ we obtain the following:

Theorem 2. *Suppose* $u(x, \cdots), u_0(x) \in X_s^{(2)}$ $(0 \le s < 1)$ *with all the other assumptions the same as Theorem 1. Then the Cauchy problem* (1)-(2) *has a unique solution in* $X_s^{(2)}$.

Remarks. The works of Baouendi-Goulaouic-Treves ([1]) and Metivier ([4]) assert the uniqueness of solutions of the Cauchy problem (1)-(2) under the assumption of integrability condition of generalized Hamilton field of F provided $u \in C^2$. In particular if the quasilinear equation (7) becomes semilinear it it enough for $u \in C^1$. However their works leave out the question of existence of solutions to (1)-(2).

To estimate the norm of $\partial_x u$ for u holomorphic, Nirenberg uses the Cauchy inequality ([7; p.575]). However we cannot use the Cauchy inequality in our case as we require that u has first order continuous derivatives. This is the major differnence between Nirenberg's ([7]) and ours.

Acknowledgement. I would like to thank F. Treves for answering my foolish questions. I would also like to thank L. Nirenberg who brought T. Nishida's work to my attention.

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