Kyungpook Mathematical Journal
Volume 32, Number 3, October, 1992.

SEMI-DIRECT SUM OF N-MODULES

Young-In Kwon

Dedicated to Professor Younki Chae on his 60th birthday

1.Introduction

Algebraic systems with addition and multiplication,but in which only one of the distributive laws is satisified,have been studied by Dickson([3]) , Zassenhaus([9]) and others ([1], [2],[4]). In particular Blackett([2]) gave a structure theory for a special classes of near-rings and Beidleman ([1]) and Scott([8]) studied the properties of near-ring and near-ring modules. In this paper we obtained some properties of a short exact sequence of near-ring modules. Throughout this paper N stands for a right near-ring. For basic results and information about near-rings see Pilz [7].

Definition 1.1. A *near-ring module* ${}_NM$, (briefly N-module M) is a pair (M, f), where $M = (M, +)$ is a group, and $f : N \times M \to M$ is a mapping,$f(n, m) = nm$ such that for all $n_1, n_2 \in N, m \in M$,

$$(n_1 + n_2)m = n_1m + n_2m$$

and

$$(n_1n_2)m = n_1(n_2m).$$

Definition 1.2. A subset A of an N-module M is an *N-submodule* of M if

(1) $(A, +)$ is a normal N-subgroup of $(M, +)$,
(2) for any $n \in N$,$a \in A$, and $m \in M$, $n(m + a) - nm \in A$.

Received March 30, 1992.

If A is an N-submodule of M, the factor group M/A can be regarded as an N-module is said to be a *factor module* by defining $n(m + A) = nm + A$. The natural group epimorphism $f : M \to M/A$ becomes an N-epimorphism.

Definition 1.3. Let $\{M_k : f_k\}$ be a collection of N-modules M_k with a corresponding collection of N-homomorphism $f_k : M_k \to M_{k+1}$.

The sequence $\cdots M_{k-1} \xrightarrow{f_{k-1}} M_k \xrightarrow{f_k} M_{k+1} \cdots$ called an *exact sequence* if $Ker f_k = Im f_{k-1}$. An exact sequence of the form

$(0) \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to (0)$

is called a *short exact sequence.*

For all N-homomorphism $f : M_1 \to M_2$, $f(M_1)$ is an N-subgroup of M_2. In general, $f(M_1)$ is not necessarily an N-submodule of M_2.

Definition 1.4. An N-homomorphism $f : M_1 \to M_2$ is *normal* if $f(M_1)$ is an N-submodule of M_2.

Definition 1.5. A short exact sequence $(0) \to M_1 \to M_2 \xrightarrow{f} M_3 \to (0)$ *almost splits* if there exists an N-homomorphism $g : M_3 \to M_2$(not necessarily normal) with $fg = I_{M_3}$, where I_{M_3} is the identity map of M_3. An exact sequence $M_1 \xrightarrow{f} M_2 \to (0)$ *almost splits* if there exists an N-homomorphism $g : M_2 \to M_1$ with $fg = I_{M_2}$. Also an exact sequence $(0) \to M_1 \xrightarrow{h} M_2$ *almost splits* if there is an N-homomorphism $g : M_2 \to M_1$ with $gh = I_{M_1}$. Such an N-homomorphism g is called an *almost splitting N-homomorphism.*

2. Semi-direct sum

Definition 2.1. An N-module M is said to be *semi-direct sum* of its N-subgroup A and B if A is an N-submodule, $M = A + B$ and $A \bigcap B = (0)$.

Here A is called to a *semi-direct summand* of M. It is denoted by $M = A \dot{+} B$.

Theorem 2.2. *For a short exact sequence* $(0) \to M_1 \xrightarrow{h} M_2 \xrightarrow{f} M_3 \to (0)$,

the followings are equivalent:

(1) *The short exact sequence* $(0) \to M_1 \xrightarrow{h} M_2 \xrightarrow{f} M_3 \to (0)$ *almost splits.*

(2) $M_2 = h(M_1)\dot{+}g(M_3)$ *where* g *is the almost splitting* N*-homomorphism for* f.

(3) *The exact sequence* $(0) \to M_1 \xrightarrow{h} M_2$ *almost splits.*

Proof. (1) $\to$ (2). If $g : M_3 \to M_2$ is the almost splitting N-homomorphism, then $fg = I_{M_3}$ and $g(M_3)$ is an N-subgroup of M_2. For all $b \in M_2$, $f(b - gf(b)) = f(b) - (fg)f(b) = f(b) - f(b) = 0$.

So $b-(gf)(b) \in Kerf$, that is, $b \in Kerf+g(M_3)$. If $b \in Kerf \bigcap g(M_3)$, $f(b) = 0$ and there exists some element c in M_3 with $g(c) = b$. Then $0 = f(b) = f(g(c)) = (fg)(c) = c$. Since g is an N-homomorphism, $b = g(c) = g(0) = 0$. Thus $Kerf \bigcap g(M_3) = (0)$ and so $M_2 = Kerf\dot{+}g(M_3)$.Since the sequence is exact at M_2, $Kerf = h(M_1)$ and so $M_2 = h(M_1)\dot{+}g(M_3)$. (2) $\to$ (3). For all $b \in M_2$, $b = h(a) + g(c)$ for some $a \in M_1$, $c \in M_3$. Define $f' : M_2 \to M_1$ by $f'(b) = a$. Then f' is well defined. For any $b_1 = h(a_1) + g(c_1)$ and $b_2 = h(a_2) + g(c_2)$ in M_2, if $b_1 = b_2$, $h(a_1) = h(a_2)$ and so $a_1 = a_2$ since h ia an N- monomorphism. Then $f'h = I_{M_1}$. (3) $\to$ (1). Let $k : M_2 \to M_1$ be such that $kh = I_{M_1}$. For all $b \in M_2$, $k(b - hk(b)) = k(b) - (kh)k(b) = 0$ and so $b - (hk)(b) \in Kerk$, that is, $b \in Kerk\dot{+}h(M_1)$. If $b \in Kerk \bigcap h(M_1)$, then $k(b) = 0$ and there exists some element $a \in M_1$ with $b = h(a)$. Thus $0 = k(b) = k(h(a)) = (kh)(a) = a$. Since h is an N-homomorphism, $b = h(a) = h(0) = 0$, that is, $h(M_1) \bigcap Kerk = (0)$. And since $h(M_1) = Kerf$, $M_2 = Kerk \dot{+} h(M_1)$. Therefore $M_3 = f(M_2) = f(Kerk)$. If $f_1 = f|Kerk$,it is an N-isomorphism. For any c in M_3, there exists some element b in $Kerk$ with $f(b) = c$, that is, $f_1(b) = c$. and so f_1 is an N-epimorphism.If b_1, b_2 in $Kerk$ with $f_1(b_1) = f_1(b_2)$, $b_1 - b_2 \in Kerk \bigcap Kerf = (0)$ and then $b_1 = b_2$. Let $g = f_1^{-1}$. We have $g(M_3) = Kerk$ and clearly $fg = I_{M_3}$.

Theorem 2.3. *If a exact sequence* $M_1 \xrightarrow{h} M_2 \to (0)$ *almost splits with an almost splitting* N*-homomorphism* $g : M_2 \to M_1$*, then* $M_1 = Kerh\dot{+}Img$.

Proof. For any $a \in M_1$, $h(a) \in M_2$ and consequently $g(h(a)) \in M_1$. Since $hg = I_{M_2}$, $h(b - g(h(b)) = h(b) - (hg)(h(b)) = 0$ and so $b \in Kerh + Img$. For $a \in Kerh \bigcap Img$, there exists an element b in M_2 such that $g(b) = a$ while $h(a) = 0$. Hence $0 = h(a) = h(g(b)) = (hg)(b) = b$ which implies $b = 0$. Since g is an N-homomorphism, Img is an N-subgroup of M_1 with $M_1 = Kerh\dot{+}Img$.

Theorem 2.4. *If* $f : M_1 \to M_2$ *is an* N*-homomorphism and* $(0) \to M_1 \xrightarrow{f} M_2$ *almost splits with an almost splitting* N*-homomorphism* $g : M_2 \to M_1$*, then* $M_2 = Kerg\dot{+}Imf$.

Proof. Since $gf = I_{M_1}$, for any $b \in M_2, g(b) \in M_1$ and $f(g(b)) \in M_2$, $g(b-(fg)(b)) = g(b)-(gf)(g(b)) = 0$ and consequently $b \in Kerg + Img$. For $b \in Kerg \bigcap Imf$, there exists an element $a \in M_1$ such that $f(a) = b$ while $g(b) = 0$. Hence $0 = g(b) = g(f(a)) = (gf)(a) = a$ which implies $a = 0$. Since f is an N-homomorphism, $b = 0$. Hence $M_2 = Kerg \dot{+} Imf$, since Imf is an N-subgroup of M_2.

Theorem 2.5. *If a short exact sequence* $(0) \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} (0)$ *almost splits at one end, it almost splits at the other hand.*

Proof. Suppose $(0) \to M_1 \xrightarrow{f} M_2$ almost spliting N-homomorphism h. By the Theorem 2.4, $M_2 = Kerh \dot{+} Imf$.

Since $Imf = Kerg$, $g(M_2) = g(Kerh) = M_3$. Furthermore $g|Kerh$ is an N-monomorphism. In fact, if $g(k_1) = g(k_2), k_1, k_2 \in Kerh$, $k_1 - k_2 \in Kerh \bigcap Kerg = (0)$ and so $k_1 = k_2$. Thus $g' = g|Kerh$ is an N-isomorphism and $(g')^{-1} : M_3 \to Kerh$ is the desired almost spliting N-homomorphism. Conversely, if $M_2 \xrightarrow{g} M_3 \to (0)$ almost splits, there is an N- homomorphism $h : M_3 \to M_2$ such that $gh = I_{M_3}$. By the Theorem 2.3, $M_2 = Kerg \dot{+} Imh$ and so for $b \in M_2, b = x+y, x \in Kerg = Imf, y \in Imh$. If we define $k : M_2 \to M_1$ by $k(b) = f^{-1}(y)$,then $kf = I_{M_1}$ and so $(0) \to M_1 \xrightarrow{f} M_2$ almost splits. For any $a \in M_1$, $(kf)(a) = f^{-1}(f(a)) = a$.

Remarks. Let A be an N-submodule of M and $i : A \to M$ the embedding N-monomorphism. If $(0) \to A \xrightarrow{i} M \to M' \to (0)$ almost splits then by the Theorem 2.2, A is a semi-direct summand of M. Conversely, if $M = A \dot{+} C$, where C is an N-subgroup of M, then each $b \in M$ has a unique representation, $b = a + c, a \in A, c \in C$ and the N-homomorphism $h : M \to A$ defined by $h(b) = a$ is an almost spliting N-homomorphism. For any $b_1, b_2 \in M, b_1$ and b_2 have a unique representation $b_1 = a_1 + c_1, b_2 = a_2 + c_2, a_1, a_2 \in A, c_1, c_2 \in C$, respectively. Then $h(b_1 + b_2) = h((a_1 + a_2) + (c_1 + c_2)) = a_1 + a_2 = h(a_1 + c_1) + h(a_2 + c_2) = h(b_1) + h(b_2)$ and $h(nb_1) = h(n(a_1 + c_1)) = h(na_1 + nc_1) = na_1 = nh(b_1)$. Also $hi(a) = h(i(a)) = h(a) = h(a + 0) = a$. Thus we have

Theorem 2.6. *A short exact sequence* $(0) \to A \xrightarrow{i} M \to M' \to (0)$ *almost splits if and only if* A *is a semi-direct summand of* M.

From now in this paper, we assume that N is a zero-symmetric right near-ring.

Definition 2.7. An N-module A is *almost projective* if every exact sequence of the form $M \to A \to (0)$ almost splits.

Since every exact sequence of the form $P \xrightarrow{f} B \to (0)$ can be embedded in a short exact sequence $(0) \to Kerf \to P \xrightarrow{f} B \to (0)$, we have

Theorem 2.8. *An N-module A is almost projective if and only if every short exact sequence of the form $(0) \to P \to B \to A \to (0)$ almost splits.*

Theorem 2.9. *An N-module A is almost projective if and only if $A \cong M/K$ implies K is a semi-direct summand of M.*

Proof. If A is almost projective and $A \cong M/K$, then $(0) \to K \xrightarrow{i} M \to A \to (0)$ almost splits and by the Theorem2.6, K is a semi- direct summand of M. Conversely, suppose we have an exact sequence $M \xrightarrow{f} A \to (0)$. Thus $A \cong M/Kerf$ and consequently $Kerf$ is a semi-direct summand of M. Using the Theorem 2.6 again we have that $(0) \to Kerf \to M \xrightarrow{f} A \to (0)$ almost splits.

Theorem 2.10. *If $M = A \oplus B$, M is almost projective, then A and B are almost projective.*

Proof. Suppose we have $P \xrightarrow{f} A \to (0)$ with $Kerf = K$. Let $P \times B$ be the Cartesian product N-module of P and B. And we define the map $g : P \times B \to A \oplus B$ by $g(p, b) = f(p) + b, p \in P$ and $b \in B$. Since the elements of A and B commute, g is an N-epimorphism and $Kerg = K^\star = K \times \{0\}$. For any (p_1, b_1) and $(p_2, b_2) \in P \times B$ $(p_1, p_2 \in P, b_1, b_2 \in B)$, $g((p_1, b_1) + (p_2, b_2)) = g(p_1 + p_2, b_1 + b_2) = f(p_1 + p_2) + (b_1 + b_2) = \{f(p_1) + f(p_2)\} + (b_1 + b_2) = \{f(p_1) + b_1\} + \{f(p_2) + b_2\} = g(p_1, b_1) + g(p_2, b_2)$. And $g(n(p_1, b_1)) = g(np_1, nb_1) = f(np_1) + nb_1 = nf(p_1) + nb_1 = n\{f(p_1) + b_1\} = ng(p_1, b_1)$. Thus g is an N-homomorphism. For any $a + b \in A \oplus B (a \in A, b \in B)$, since f is surjective, then exists some elements p in P with $f(p) = a$. Then $g(p, b) = f(p) + b = a + b$. Next if $g(p, b) = f(p) + b = 0$, $f(p) = 0$ and $b = 0$, that is, $p \in Kerf = K$ and $b = 0$. Then $(p, b) \in K \times \{0\}$. Conversely if $(p, b) \in K \times \{0\}$, then $p \in K$ and $b = 0$ and so $f(p) = 0$ and $b = 0$. Thus $g(p, b) = f(p) + b = 0$. We have $(p, b) \in Kerg$. Since $A \oplus B = (P \times B)/Kerg$, by the Theorem 2.9, $K^\star = Kerg$ is a semi-direct summand of $P \times B$. Then $P \times B = K^\star \dot{+} Q$ for some N-subgroup Q of $P \times B$.Now $P^\star = P \times \{0\}$ is an N-submodule of $P \times B$ and so every $p \in P^\star$ has a unique representation, $p = q + k$ where $q \in Q, k \in K^\star$. Since $K^\star$ is an N-submodule of $P^\star$,$q (= p - k) \in P^\star$ and

$q \in X = P^* \bigcap Q$, that is,$P^* = K^* + X$.Thus K is a semi-direct summand of P and $(0) \to K \to P \to A \to (0)$ almost splits which shows that A is almost projective. Similarily B is also almost projective.

Remark. Supposing that the direct sum of N-subgroups is an N-subgroup, we can prove the converse of the above theorem.

References

[1] Beidleman,J.C., *On near-rings and near-ring modules* , Doctoral dissertation, Pensyv. State University, 1964.

[2] Blackett,D.W., *Simple and semi-simple near-rings*, Proc. Amer. Math. Soc.,4 :772-785,1953.

[3] Dickson, L.E., *Definitions of a group and a field by independent postulates*, Tran. Amer. Math. Soc., 6 :198-204, 1905.

[4] Fröhlich, A., *Distributively generated near-rings I*, Ideal theory, Proc. London Math. Soc., 8: 76-94, 1958.

[5] ________, *Distributively generated near-rings II*, Representation theory, Proc. London Math. Soc., 8: 95-108, 1958.

[6] Oswald, A., *Some topics in the structure theory of near-rings*, Doctoral Thesis, Univ. of York, 1973.

[7] Pilz, G., *Near-rings*, North Holland, New York, 1983.

[8] Scott, S.D., *Near-rings and near-ring modules*, Doctoral dissertation,Australian National University, 1970.

[9] Zassenhaus, H., *Uber endliche Fastkorper*, Abh. Math.Sem.Univ. Hamburg II, 187-320, 1935/36.

Department of Mathematics, Gyeongsang National University, Chinju 660-701, Korea.