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A POSITIVE MEASURE CHARACTERIZED BY AN INNER FUNCTION

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Dedicated to Professor Younki Chae on the occasion of his sixtieth birthday

1. Introduction

Let U be the open unit disc in the complex plane. Let T be the boundary of U identified with $[0, 2\pi]$. We, in this note, are interested in the positive Borel measure μ satisfying

$$\mu(S_{h,\theta}) \leq Ch^{1-p} \tag{1.1}$$

for some positive constant C (independent of h), where

$$S_{h,\theta} = \{z = re^{it} : 1 - h < r < 1, |\theta - t| < \frac{h}{2}\}.$$

P. Ahern and M. Jevtic characterized those measures satisfying (1.1) in terms of mean growth conditions of holomorphic functions :

Theorem A[AJ1. Theorem 1]. *Let $0 < p < 1$. Let μ be a positive Borel measure on U. Then the followings are equivalent :*

(1) *There is $C = C(p)$ such that*

$$\mu(S_{h,\theta}) \leq Ch^{1-p}$$

for all $S_{h,\theta}$.

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(2) *There is* $C = C(p)$ *such that*

$$\int_U |f(z)|^p d\mu(z) \leq C \int_U |f'(z)|^p (1-|z|)^{p-1} dxdy$$

for all holomorphic f *with* $\int_U |f'(z)|^p(1-|z|)^{p-1}dxdy < \infty$.

(3) *There is* $C = C(p)$ *such that*

$$\int_U |f(z)|^p d\mu(z) \leq C \int_U |D^{1+p} f| dxdy$$

for all holomorphic f *with* $\int_U |D^{1+p}f|dxdy < \infty$.

Here and after, $D^p f$, $0 < p < \infty$, denotes the fractional derivative of order p of f defined, for $f(z) = \sum a_k z^k$ holomorphic in U, by

$$D^p f(z) = \sum (k+1)^p a_k z^k, \quad z \in U.$$

Also, when $0 < p < \infty$, the classical Hardy space norm

$$\sup_{0 \leq r < 1} \left(\int_T |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}$$

of f holomorphic in U will be denoted by $\|f\|_p$. It is well-known that if $\|f\|_p < \infty$ then the non-tangential maximal function $Nf(\theta)$ (See [DU] or [GA]) satisfies $\int_T (Nf(\theta))^p \, d\theta < \infty$. For $0 < q < 1$ and $E \subset T$, the q-dimensional Hausdorff measure (on T) is defined by

$$H^q(E) = \inf\{\sum |I_n|^q : E \subset \bigcup I_n,\ I_n \text{are open arcs}\}.$$

It is known [AD] that

$$\int_0^\infty H^{1-p}(\{Nf > t\})\, dt \leq C \|D^p f\|_1.$$

if $p < 1$.

By an inner function we mean a bounded holomorphic function f defined on U for which

$$\lim_{r \to 1} |f(re^{i\theta})| = 1, \quad a.e.\, \theta \in T.$$

Given a nonzero sequence $\{\alpha_n\}$ of complex numbers in U, with $\sum(1 - |\alpha_n|) < \infty$, and for a nonnegative integer integer k , the product

$$B(z) = z^k \prod_{n=1}^{\infty} \frac{\alpha_n - z}{1 - \bar{\alpha_n} z} \frac{|\alpha_n|}{\alpha_n}, \quad z \in U,$$

defines an inner function called the Blaschke product. When $\frac{1}{2} < p < 1$, there are a lot of interesting results related to the p-th mean growth of the derivatives of inner functions (See, for example, [AH], [AJ1], [AJ2]).

2. Results

Our result in this note is the following characterization of a measure satisfying (1.1) in terms of the growth conditions on inner functions :

Theorem. *Let $\frac{1}{2} < p < 1$. Then the following (1), (2), and (3) are equivalent.*

(1) *There is $C = C(p)$ such that*

$$\mu(S_{h,\theta}) \leq Ch^{1-p}$$

for all $S_{h,\theta}$.

(2) *There is $C = C(p)$ such that*

$$\int_U |f(z)| d\mu(z) \leq C \|D^p f\|_1$$

for all holomorphic f with $\|D^p f\|_1 < \infty$.

(3) *There is $C = C(p)$ such that*

$$\int_U |B(z)| d\mu(z) \leq C \left(\|B\|_p^p + \|B'\|_p^p \right)$$

for all Blaschke product B with $\|B'\|_p < \infty$.

3. Proof of Theorem

(1) $\Rightarrow$ (2): Suppose (1). Then by a result of Ahern and Jevtić(See [AJ1. Proof of Theorem 1])

$$\mu(\{|f| > t\}) \leq CH^{1-p}(\{Nf > t\}).$$

Hence

$$\int |f(z)| d\mu(z) = \int_0^\infty \mu(\{|f| > t\})\, dt \leq C \int_0^\infty H^{1-p}(\{Nf > t\})\, dt.$$

(2) then follows from the strong Hausdorff capacity estimates of D. Adams [AD] :

$$\int_0^\infty H^{1-p}(\{Nf > t\})\, dt \leq C \|D^p f\|_1.$$

(2) $\Rightarrow$ (3): Suppose (2). Then

$$\int |B| d\mu(z) \leq C \|D^p B\|_1 \tag{3.1}$$

for all Blaschke product B with $\|D^p B\|_1 < \infty$. Let

$$g(z) = zB(z), \quad \text{and} \quad f(z) = zB'(z), \quad z \in U.$$

Then

$$g'(z) = B(z) + f(z), \quad z \in U, \tag{3.2}$$

and

$$D^p B(z) = \frac{1}{\Gamma(1-p)} \int_0^1 g'(tz) \left(\log \frac{1}{t}\right)^{-p} dt. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned} |D^p B(re^{i\theta})| \ &\leq \ \frac{1}{\Gamma(1-p)} \int_0^1 |B(tre^{i\theta})| \left(\log \frac{1}{t}\right)^{-p} dt \\ &+ \ \frac{1}{\Gamma(1-p)} \int_0^1 |f(tre^{i\theta})| \left(\log \frac{1}{t}\right)^{-p} dt. \end{aligned} \tag{3.4}$$

Since $|B| \leq 1$,

$$\begin{aligned} &\int_0^{2\pi} \int_0^1 |B(tre^{i\theta})| \left(\log \frac{1}{t}\right)^{-p} dt d\theta \\ \leq \quad & 2\pi\Gamma(1-p)\|B\|_1 \leq 2\pi\Gamma(1-p)\|B\|_p^p. \end{aligned} \tag{3.5}$$

On the other hand, since

$$\int_0^{2\pi} |f(tre^{i\theta})| d\theta \leq \int_0^{2\pi} |f(te^{i\theta})| d\theta$$

if $t < 1$, we have

$$\int_0^{2\pi} \int_0^1 |f(tre^{i\theta})| \left(\log \frac{1}{t}\right)^{-p} dt d\theta \leq \int_0^1 \left(log \frac{1}{t}\right)^{-p} dt \int_0^{2\pi} |f(te^{i\theta})| dt. \tag{3.6}$$

We now estimate the last quantity of (3.6). Remind the Hardy-Stein idenity [ST] :

$$t \frac{d}{dt} \int_0^{2\pi} |f(te^{i\theta})|^q d\theta = q^2 A(q, f, t), \quad 0 < t < 1, \tag{3.7}$$

where

$$A(q,f,t)=\int\int_{tU}|f(z)|^{q-2}|f'|^2dxdy.$$

Since $f(0)=0$, integrating $q=1$ case of (3.7) we obtain

$$\int_0^{2\pi}|f(te^{i\theta})|d\theta=\int_0^t A(1,f,s)\frac{ds}{s}. \tag{3.8}$$

By the Schwarz-Pick's Lemma [GA],

$$|f(z)|=|z||B'(z)|\le|z|(1-|z|)^{-1}\le(1-|z|)^{-1},\quad z\in U,$$

so that we have

$$|f(z)|^{-1}\le|f(z)|^{p-2}(1-|z|)^{p-1}. \tag{3.9}$$

From (3.8) and (3.9),

$$\int_0^{2\pi}|f(te^{i\theta})|d\theta\le\int_0^t A(p,f,s)\frac{(1-s)^{p-1}}{s}ds.$$

Hence

$$\begin{aligned}&\int_0^1\left(\log\frac{1}{t}\right)^{-p}dt\int_0^{2\pi}|f(te^{i\theta})|d\theta\\ \le\ &\int_0^1\left(\log\frac{1}{t}\right)^{-p}dt\int_0^t A(p,f,s)\frac{(1-s)^{p-1}}{s}ds\\ =\ &\int_0^1\frac{(1-s)^{p-1}}{s}ds\int_s^1 A(p,f,s)\left(\log\frac{1}{t}\right)^{-p}dt.\end{aligned} \tag{3.10}$$

Noting that

$$\int_s^1\left(\log\frac{1}{t}\right)^{-p}dt\le\int_s^1(1-t)^{-p}dt=\frac{(1-s)^{1-p}}{1-p},$$

we have from (3.10),

$$\int_0^1\left(log\frac{1}{t}\right)^{-p}dt\int_0^{2\pi}|f(te^{i\theta})|d\theta\le\frac{1}{1-p}\int_0^1 A(p,f,s)\frac{ds}{s}. \tag{3.11}$$

Integrating (3.7) once more with p in place of q,

$$p^2\int_0^1 A(p,f,s)\frac{ds}{s}=\lim_{\rho\to1}\int_0^{2\pi}|f(\rho e^{i\theta})|^p d\theta. \tag{3.12}$$

If we put (3.6), (3.11), and (3.12) together, we obtain

$$\int_0^{2\pi}\int_0^1 |f(tre^{i\theta})| \left(\log\frac{1}{t}\right)^{-p} dt d\theta \le \frac{2\pi}{p^2(1-p)}\|f\|_p^p. \tag{3.13}$$

Now, it follows from (3.4), (3.5), (3.13), and the fact $\|f\|_p = \|B'\|_p$ (See [DU] for example) that

$$\begin{aligned}\int_T |D^p B(re^{i\theta})|\frac{d\theta}{2\pi} &\le \|B\|_p^p + \frac{1}{p^2\Gamma(2-p)}\|f\|_p^p \\ &\le C\left(\|B\|_p^p + \|B'\|_p^p\right)\end{aligned}$$

for all $r : 0 < r < 1$. Therefore by (3.1)

$$\|D^p B\|_1 \le C\left(\|B\|_p^p + \|B'\|_p^p\right).$$

(3) $\Rightarrow$ (1): Suppose (3) and denote $S_{h,\theta}$ by S_h for simplicity. Since $\mu(U) < \infty$ by (3), we may assume that $h < \frac{1}{2}$. Set $\rho = 1 - 2h$ and $\alpha = \rho e^{i\theta}$. Let

$$B(z) = \frac{z-\alpha}{1-\bar{\alpha}z}, \quad z \in U.$$

Then for $z \in S_h$ we have

$$\begin{aligned}|1-\bar{\alpha}z| &\le |e^{i\theta} - \rho(1-h)e^{i(\theta+h/2)}| \\ &= |1 - 2\rho(1-h)cos\frac{h}{2} + \rho^2(1-h)^2|^{1/2} \\ &\le \left[1 - 2\rho(1-h)(1-\frac{h^2}{8}) + \rho^2(1-h)^2\right]^{1/2} \\ &= h\left[(2-2h)^2 + \frac{(1-2h)(1-h)}{4}\right]^{1/2} \\ &\le \frac{\sqrt{37}}{2h},\end{aligned}$$

so that

$$|B(z)| \ge \frac{2}{\sqrt{37}}. \tag{3.14}$$

Also, for $z \in S_h$,

$$|B(z)| \le \frac{2h}{|1-\bar{\alpha}z|} \le 4h. \tag{3.15}$$

On the other hand, because $\frac{1}{2} < p < 1$, we have

$$\begin{aligned} \|B'\|_p^p &= \sup_z \int_T \left[\frac{1-|\alpha|^2}{|1-\bar{\alpha}z|^2}\right]^p \frac{|dz|}{2\pi} \\ &= (1-\rho^2)^p \left[\int_0^{2\pi} |1-\rho e^{i\theta}|^{-2p} \frac{d\eta}{2\pi}\right] \qquad (3.16) \\ &\le C(1-\rho)^{1-p} \le Ch^{1-p}. \end{aligned}$$

Now, (3.14),the hypothesis (3), (3.15), (3.16), and the fact $\frac{1}{2} < p$, in this order, leads us to

$$\begin{aligned} \mu(S_h) = \int_{S_h} d\mu &\le \frac{\sqrt{37}}{2} \int_{S_h} |B(z)| d\mu(z) \\ &\le C\left(\|B\|_p^p + \|B'\|_p^p\right) \\ &\le C(h^p + h^{1-p}) \le Ch^{1-p}. \end{aligned}$$

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