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SOME PROPERTIES OF ADMISSABLE SOLUTIONS FOR A SYSTEM OF CONSERVATION LAWS OF MIXED TYPE

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Dedicated to Sixtieth Birthday of Professor Younki Chae

1. Introduction

In this paper we consider the system of conservation laws

$$u_t - v_x = 0, \quad v_t - \sigma(u)_x = 0. \tag{1.1}$$

Here σ is a smooth function that is monotonically increasing except in an interval $[\alpha, \beta]$. This system consitutes a model for phase transitions in a van der Waals gas [1,2,4,5,$\cdots$,9] and in elastic plastic rods [3].

This system (1.1) is hyperbolic in the regions

$$D = \{(u,v) | u \leq \alpha \text{ or } u \geq \beta\}$$

and elliptic in the region

$$E = \{(u,v) | \alpha < u < \beta\}.$$

Such a mixed type was studied by James [3], Hattori [1], [2], Shearer [5], [6], [7] and Slemrod [8], [9]. The study of possible criterion for the admissibility of shock waves was begun by James and Slemrod [3], [8]. In [8] Slemord presented admissibility criteria for weak solutions of the equations

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governing isothermal motion of a van der Waals fluid. He dicussed two admissibility criteria: a standard viscosity criterion and a vicosity-capillarity criterion. The former is discarded: it rules out propagating shock waves near the equilibrium co-existence line. The latter allows for propagating shock waves to exists. In [7] Shearer characterizes those phase transitions satisfying the viscosity-capillarity admissibility criteria of Slemrod and lying near the Maxwell line. He stated the important of the condition (C) conjectured by Slemrod. We shall describe the condition (C) which is possible to analyze the Riemann problem completely.

2. Preliminaries

We consider the system of conservation laws

$$u_t - v_x = 0, \quad v_t - \sigma(u)_x = 0, \tag{2.1}$$

where $\sigma : R \to R$ is of class $C^r (r \geq 2)$ and has the following property (P).
(P) There exist numbers $\alpha < \eta < \beta$ such that

$$\sigma'(u) \geq \text{ for } u \notin (\alpha, \beta), \sigma'(u) < 0 \text{ for } u \in (\alpha, \beta)$$

$$sgn\sigma''(u) = sgn(u - \eta).$$

Define $\gamma < \alpha$ and $\delta > \beta$ by $\sigma(\gamma) = \sigma(\beta)$ and $\sigma(\alpha) = \sigma(\delta)$. Let $m < M$ be given by $\sigma(m) = \sigma(M)$ and

$$\int_m^M [\sigma(u) - \sigma(m)]du = 0.$$

If a state (u_1, v_1) may be joined to state (u_2, v_2) by a shock wave $x = x(t)$ with constant shock speed $s = \dot{x}(t)$, then the Rankine-Hugoniot jump conditions [4]

$$v_2 - v_1 = -s = (u_2 - u_1) \tag{2.2}$$

$$\sigma(u_2) - \sigma(u_1) = -s(v_2 - v_1) \tag{2.3}$$

hold. It follows from (2.2) and (2.3) that

$$s^2 = \frac{\sigma(u_2) - \sigma(u_1)}{u_2 - u_1} \tag{2.4}$$

holds.

A shock wave $\{(u_1, v_1), (u_2, v_2), s\}$ is admissible according to the viscosity-capillarity criterion [7] and [8] if there is a travelling wave solution

$$(u, v) = (u(\xi), v(\xi)), \quad \xi = (x - st)/\epsilon \tag{2.5}$$

of the system

$$u_t - v_x = 0, \quad v_t - \sigma(u)_x = \epsilon v_{xx} - A\epsilon^2 u_{xxx} \tag{2.6}$$

(A constant, $0 < A \leq \frac{4}{3}$)
with boundary conditions

$$\begin{aligned} (u, v)(-\infty) &= (u_1, v_1)(u, v)(+\infty) = (u_2, v_2), \\ (u_x, v_x)(\pm\infty) &= (0, 0). \end{aligned} \tag{2.7}$$

From (2.6) it follows that (u, v) satisfies

$$-s\frac{du}{d\xi} = \frac{dv}{d\xi}, \quad -s\frac{dv}{d\xi} = \frac{d}{d\xi}(\sigma(u) + \frac{dv}{d\xi} - A\frac{d^2u}{d\xi^2}). \tag{2.8}$$

Integration of (2.8) from $-\infty$ to ξ and use of a subsitution show that a solution u of (2.6), (2.7) must satisfy the second order ordinary differential equation

$$A\frac{d^2u}{d\xi^2} = -s\frac{du}{d\xi} + \sigma(u) - \sigma(u_1) - s^2(u - u_1) \tag{2.9}$$

with boundary conditions

$$u(-\infty) = u_1, u(+\infty) = u_2, \frac{du}{d\xi}(\pm\infty) = 0. \tag{2.10}$$

We say that $u_1 \to u_2$ is a connection with speed s if there is a solution of (2.9) and (2.10).

A shock wave is admissible if it is the limit of travelling wave solutions of (2.6). A piecewise smooth solution of (2.1) is admissible if all its shock waves are admissible.

If $u_1 \to u_2$ is a connection with speed s, then for any v_1, $\{(u_1, v_1), (u_2, v_2), s\}$ represents an admissible shock, where v_2 is given by (2.2). Note that if $u_1 \to u_2$ is a connection with speed s, then $u_2 \to u_1$ is a connection with speed $-s$.

Let $u_1 < \alpha$ and $u_2 > \beta$. With s^2 given by (2.4), suppose that $s^2 < \sigma'(u_k)(k = 1, 2)$, and take $s \geq 0$. Then $(u_k, 0)(k = 1, 2)$ are saddle points.

A solution of (2.9), (2.10) consists of the unstable manifold of $(u_1, 0)$ joined to the stable manifold of $(u_2, 0)$. Let $\Gamma_1(u_1, u_2)$ denote the connected component of the unstable manifold of $(u_1, 0)$ in the upper half plane $\dot{u} = \frac{du}{d\xi} \geq 0$ and containing $(u_1, 0)$. Similarly, the connected component of the stable manifold of $(u_2, 0)$ in the upper half plane, and containing $(u_2, 0)$, will be denoted by $\Gamma_2(u_1, u_2)$. Then $\alpha > u_1 \to u_2 > \beta$ is saddle connection with speed $s \geq 0$ given by (2.4) if and only if $s^2 < \sigma'(u_k), k = 1, 2$, and $\Gamma_1(u_1, u_2) = \Gamma_2(u_1, u_2)$. We may assume that Γ_1 and Γ_2 are parametrized by $u : \frac{du}{d\xi} = w_k(u)$ on $\Gamma_k(u_1, u_2)$. Then (2.9) may be written

$$(2.11) \qquad Aw_k \frac{dw_k(u)}{du} = -sw_k(u) + \sigma(u) - \sigma(u_1) - s^2(u - u_1).$$

Lemma 2.1([9], [10]). *Let $u_1 \in [M, \delta]$. There exists one $u_2 = \hat{u}_2(u_1) < u_1$ such that $u_1 \to u_2$ is a saddle-saddle connection with nonnegative speed. The corresponding speed $\hat{s}(u_1)$ is positive for $u_1 > M$. For each $u_1 \in [M, \delta], \hat{u}_2(u_1) > \gamma$ and $\hat{u}_2$ is of class C^{r-1} on (M, δ).*

Let $u_1 \in [M, \delta]$ be fixed and let $u_2 = u_2(s) < \alpha$ be given by

$$(2.12) \qquad s^2 = \frac{\sigma(u_1) - \sigma(u_2)}{u_1 - u_2}$$

for each s, $0 \leq s < \sqrt{\sigma'(u_1)}$. Then $(u_1, 0)$ and $(u_2, 0)$ are saddle points. The chord joining $(u_1, \sigma(u_1))$ to $(u_2, \sigma(u_2))$ cuts the graphs of σ at a third points $(u_0, \sigma(u_0))$. Moreover the trajectories Γ_1 and Γ_2 both cross the line $u = u_0$. Set $\bar{w}_k(s) = w_k(u_0(s))$. Note that $w_k(u)$ also depends on s implicitly.

Lemma 2.2([7]). *With the above notation*

$$\frac{d}{ds}[\bar{w}_1(s) - \bar{w}_2(s)] < 0$$

if $0 < s < \sqrt{\sigma'(u_)}$.

Theorem 2.1([7]). *There exists $\theta > 0$ such that if $M < u_1 < M + \theta$, then the corresponding $u_2 = \hat{u}_2(u_1)$ giving the saddle-saddle connection with speed $\hat{s}(u_1) > 0$ satisfies $u_2 > m$.*

In fact, for $u_1 \in [M, \delta]$ define $\bar{u}_2(u_1) < \alpha$ by

$$\int_{\bar{u}_2}^{u_1} [\sigma(\eta) - \sigma(u_1) - \bar{s}^2(\eta - u_1)]d\eta = 0$$
$$\bar{s}_2(u_1 - \bar{u}_2) = \sigma(u_1) - \sigma(\bar{u}_2)$$

and let $s_\gamma = s_\gamma(u_1) > 0$ be given by

$$s_\gamma^2(u_1 - \gamma) = \sigma(u_1) - \sigma(\gamma).$$

Define $\hat{u}_2 = \max\{\bar{u}_2, \gamma\}$, $\hat{s} = \min\{\bar{s}, s_\gamma\}$. Then $\hat{u}_2$ and $\hat{s}$ satisfy the result. Moreover $\hat{u}_2$ is continuous up to M.

Corollary 2.1([7]). $\hat{u}_2(u_1) \to m$ $u_1 \to M+$.

Let $\bar{u}_1 \in [M, \delta]$. Write $\bar{u}_2 = \hat{u}_2(\bar{u}_1)$, so that $\bar{u}_1 \to \bar{u}_2$ is a saddle-saddle connection with speed $\bar{s} = \hat{s}(\bar{u}_1) \geq 0$. Then $\bar{\Gamma} = \Gamma_1(\bar{u}_1, \bar{u}_2) = \Gamma_2(\bar{u}_1, \bar{u}_2)$ is the trajectory joining $(\bar{u}_1, 0)$ to $(\bar{u}_2, 0)$. Let $\bar{\Gamma} = \{\bar{w}(u) | \bar{u}_2 \leq u \leq \bar{u}_1\}$.

3. Main Theorem

In this section, we shall prove the monotonicity of $\hat{u}_2$. We need some Lemmas.

Lemma 3.1([7]). *$\hat{s}(u_1)$ is strictly monotonically increasing on $[M, \delta]$.*

Lemma 3.2. *Suppose $\int_{\bar{u}_2}^{\bar{u}_1} \bar{w}(u)du > 2\bar{s}\int_{\bar{u}_2}^{u_1} (\bar{u}_2 - u)du$. Then there exists $\mu > 0$ such that for $\bar{u}_1 < u_1 < \bar{u}_1 + \mu$, $\Gamma_1(u_1, \bar{u}_2)$ intersects Γ.*

Proof. Set $w = \frac{du}{d\xi}$ and $f(s, u) = \sigma(u) - \sigma(u_1) - s^2(u - u_1)$. From (2.9) we have

$$Aw\frac{dw}{du} = -sw + f(s, u). \tag{3.1}$$

Since $\bar{w}(\bar{u}_2) = 0$, (3.1) implies

$$\bar{s}\int_{\bar{u}_2}^{\bar{u}_1} \bar{w}(u)du = \int_{\bar{u}_2}^{\bar{u}_1} [\sigma(u) - \sigma(\bar{u}_2) - s^2(u - u_2)]du. \tag{3.2}$$

For $u_1 > \bar{u}_1$, let $\Gamma_1(u, \bar{u}_2) = \{w_1(u) | u < u_1\}$.

It suffices to show that if $u_1 > \bar{u}_1$ is chosen enough to $\bar{u}_1$, then there exists $u \in [\bar{u}_2, \bar{u}_1]$ such that

$$w_1(u) < \bar{w}(u). \tag{3.3}$$

Suppose that $w_1(u) \geq \bar{w}(u)$ for all $u \in [\bar{u}_2, \bar{u}_1]$. Then $w_1(\bar{u}_2)$ is defined and

$$(3.4)\quad A\frac{[w_1(\bar{u}_2)]^2}{2} = -\int_{\bar{u}_2}^{u_1} w_1(u)du + \int_{\bar{u}_2}^{u_1} [\sigma(u) - \sigma(\bar{u}_2) - s^2(u - \bar{u}_2)]du$$

where $s > \bar{s}$ depends on u_1 through (2.4), with $u_2 = \bar{u}_2$. Substituting (3.2) from (3.4) leads to

$$\begin{aligned} A\frac{[w_1(\bar{u}_2)]^2}{2} &= (s-\bar{s})\int_{\bar{u}_2}^{\bar{u}_1}[(s+\bar{s})(\bar{u}_2 - u) - \bar{w}(u)]du \\ &\quad + s\int_{\bar{u}_2}^{\bar{u}_1}(\bar{w}(u) - w_1(u))du + (s^2 - \bar{s}^2)\int_{\bar{u}_1}^{\bar{u}_1}(\bar{u}_2 - u)du \\ &\quad + \int_{\bar{u}_1}^{u_1}[\sigma(u) - \sigma(\bar{u}_2) - \bar{s}^2(u - \bar{u}_2) - sw_1(u)]du \\ &= (s-\bar{s})I + II + III + IV. \end{aligned}$$

Since $\bar{w}(u) \leq w_1(u)$ and $\bar{w}(u) < w_1(u)$ near $\bar{u}_1$, it follows that $II < 0$. Moreover $IV < 0$. Therefore,

$$(3.5)\qquad \frac{A[w_1(\bar{u}_2)]^2}{2} < (s-\bar{s})G(u_1)$$

where $G(u_1) = \int_{\bar{u}_2}^{\bar{u}_1}[(s+\bar{s})(\bar{u}_2 - u) - \bar{w}(u)]du + (s+\bar{s})\int_{\bar{u}_1}^{u_1}(\bar{u}_2 - u)du$.

Now G is continuous and

$$G(\bar{u}_1) = \int_{\bar{u}_2}^{\bar{u}_1}[2\bar{s}(u_2 - u_1) - \bar{w}(u)]du < 0$$

by assumption. Therefore there exists $\mu > 0$ such that if $\bar{u}_1 < u_1 < \bar{u}_1 + \mu$, then $G(u_1) < 0$. From (3.5) and $s = s(u_1) > \bar{s}$, the required contradiction is obtained and $w_1(u) < \bar{w}(u)$ for some $u \in [\bar{u}_2, \bar{u}_1]$. This completes the proof.

Lemma 3.3([6],[7]). *If $\bar{u}_1 \to \bar{u}_1$ is a saddle-saddle connection, then the corresponding trajectory $\bar{\Gamma} = \{\bar{w}(u) | \bar{u}_2 \leq u \leq \bar{u}_1\}$ joining $(\bar{u}_1, 0)$ to $(\bar{u}_2, 0)$ is concave: $\frac{d^2\bar{w}}{du^2}(u) < 0$ for $\bar{u}_2 \leq u \leq \bar{u}_1$.*

Lemma 3.4. *$\hat{s}(u_1) \to 0$ if and only if $u_1 \to M+$.*

Proof. Let $u_1 \in [M, \delta]$. Theorem 2.1 shows that there exists exactly one $u_2 = \hat{u}_2(u_1) < u_1$ such that $u_1 \to u_2$ is a saddle-saddle connection with non-negative speed $\hat{s}(u_1)$. The definition of $\hat{u}_2$ and $\hat{s}$ satisfies the following conditions

$$(3.6)\qquad \int_{\hat{u}_2}^{u_1}[\sigma(\eta) - \sigma(u_1) - \hat{s}^2(\eta - u_1)]d\eta = 0$$

$$\hat{s}^2(u_1 - \hat{u}_2) = \sigma(u_1) - \sigma(\hat{u}_2). \tag{3.7}$$

If $u_1 \to M+$, then $\hat{u}_2(u_1) \to m$ (by Corollary 2.1). Then the left hand side of (3.7) vanishes. Thus we obtain the desired result. Conversely, if $\hat{s}(u_1) \to 0$, then (3.5) and (3.6) imply that

$$\int_{\hat{u}_2}^{u_1} [\sigma(\eta) - \sigma(u_1)]d\eta = 0$$

and

$$\sigma(u_1) - \sigma(\hat{u}_2) = 0$$

hold. Since $u_1 \neq \hat{u}_2$, it follows from conditions of σ that $u_1 \to M+$ and $u_2 \to m$.

Theorem 3.1. *There exists $\varrho > 0$ such that $\hat{u}_2$ is a monotonically increasing function of $u_1 \in [M, M + \varrho)$.*

Proof. The trajectories $\hat{\Gamma} = \Gamma_k(\bar{u}_1, \hat{u}_2(u_1))(k = 1, 2)$ depend continuously upon $\bar{u}_1$ so does the shock speed $\bar{s} = \hat{s}(u_1)$. By Lemma 3.4, for $u_1 = M$, $u_2 = m$, $s = 0$, we have

$$\int_m^M \bar{w}(u)du = \int_m^M 2[\frac{\sigma(u) - \sigma(m)}{A}]^{\frac{1}{2}} du > 0.$$

Therefore, there exists $\rho > 0$ such that

$$\int_{\bar{u}_2}^{\bar{u}_1} [\bar{w}(u) - 2\bar{s}(\bar{u}_2 - u)]du > 0 \tag{3.8}$$

when $\bar{u}_1 \in [M, M + \varrho)$. But (3.8) is the hypothesis of Lemma 3.2. Consequently, for each such $\bar{u}_1$, there exists $\mu > 0$ such that $\Gamma(u_1, \bar{u}_2)$ intersects $\bar{\Gamma}$ if $\bar{u}_1 < u_1 < \bar{u}_1 + \mu$. Any intersection is locally isolated. In fact, it is easy to show it is transversal unless $\bar{w}(u) = -(s + \bar{s})(u_2 - u)$ ($s > 0$ is given by (2.4) with $u_2 = \bar{u}_2$), which occurrs at precisely one point on $\bar{\Gamma}$, by the concavity of Γ (by Lemma 3.3). Suppose $\Gamma_1(u_1, \bar{u}_2)$ does not cross the u-axis to the right of $(\bar{u}_2, 0)$. Then (3.3) implies those exist at least two interactions of $\Gamma_1(u_1, \bar{u}_2)$ and with Γ. Since $s > \bar{s}$, it is easy to check that $\frac{dw_1(\bar{u}_2)}{du} > \frac{d\bar{w}(\bar{u}_2)}{du}$ if $w_1(\bar{u}_2) = 0$. In any case $w_1(u) > \bar{w}(u)$ for $u > \bar{u}_2$ near u_2. As a result, there exist $u^* < u^{**}$ between $\bar{u}_1$ and $\bar{u}_2$ such that $w_1(u^*) = \bar{w}(u^*)$, $w_1(u^{**}) = \bar{w}(u^{**})$, $w_1(u) < \bar{w}(u)$ for $u^* < u < u^{**}$ and $w_1(u) \geq \bar{w}(u)$ for $\bar{u}_2 \leq u \leq u^*$. This implies that

$$\frac{dw_1}{du}(u^*) \leq \frac{d\bar{w}}{du}(u^*) \tag{3.9}$$

and

$$\frac{dw_1}{du}(u^{**}) \le \frac{d\bar{w}}{du}(u^{**}). \tag{3.10}$$

Note that $w = w_1$ satisfies

$$Aw\frac{dw}{du} = -sw + \sigma(u) - \sigma(\bar{u}_2) - s^2(u - \bar{u}_2) \tag{3.11}$$

and $w = \bar{w}$ satisfies (3.11) also, with $s = \bar{s}$. Therefore, when $\bar{w} = w_1$,

$$A\bar{w}\frac{d(w_1 - \bar{w})}{du} = -(s - \bar{s})\bar{w} - (s^2 - \bar{s}^2)(u - \bar{u}_2). \tag{3.12}$$

Set $F(u) = -(s - \bar{s})\bar{w} - (s^2 - \bar{s}^2)(u - \bar{u}_2)$. Then $F(\bar{u}_2) = 0$ and

$$\frac{dF}{du}(\bar{u}_2) = -(s - \bar{s})\frac{d\bar{w}}{du}(\bar{u}_2) - (s^2 - \bar{s}^2) > 0$$

since

$$\begin{aligned} \frac{d\bar{w}}{du}(\bar{u}_2) &= -(\bar{s} + [\bar{s}^2(1 - 4A) + 4A\sigma'(\bar{u}_2)]^{\frac{1}{2}})/2A \\ &< -2\bar{s} - 2\sigma'(\bar{u}_2)^{\frac{1}{2}} < -2(\bar{s} + s). \end{aligned}$$

Thus

$$F(u) < 0 \text{ for } \bar{u}_2 < u \text{ near } \bar{u}_2. \tag{3.13}$$

From (3.9), (3.10) and (3.12), it follows that

$$F(u^*) \le 0 \text{ and } F(u^{**}) \ge 0. \tag{3.14}$$

(3.13) and (3.14) imply that there exist $u_3, u_4, u^* < u_3 < u^{**} < u^4 < \bar{u}_1$ such that

$$\frac{d\bar{w}}{du}(u_3) = -(s + \bar{s}) = \frac{d\bar{w}}{du}(u_4)$$

which contradicts the concavity of $\bar{\Gamma}$ guaranted by Lemma 3.3. Hence $\Gamma_1(u_1, \bar{u}_2)$ must cross the u-axis to the right of $(\bar{u}_2, 0)$ and $\Gamma_2(u_1, \bar{u}_2)$ lies above $\Gamma_1(u_1, \bar{u}_2)$. By Lemma 2.2, $\Gamma_2(u_1, \bar{u}_2)$lies above $\Gamma_1(u_1, u_2)$ for $u_2 > \bar{u}_2$ so that $\hat{u}_2(u_1)$ must be less than $\bar{u}_2 = \bar{u}_2(u_1)$. The proof is complete.

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