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THE LLN FOR PRODUCT PROCESSES UNDER SMOOTH BOUNDARY CONDITIONS INDEXED BY SETS

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Dedicated to Professor Younki Chae on his 60th birthday

1.Introduction

In this paper, we will state and prove the uniform strong law of large numbers for a sequence of set-indexed product Poisson processes and that of set-indexed product partial sum prosesses under the 'smooth boundary condition' on the index families. This condition was invented and used to prove the same question for set-indexed partial sum processes in Bass and Pyke(1984).

Let Y_1 and Y_2 be Poisson processes with integer parameters λ_1 and λ_2 on $\mathcal{B}(\mathbf{I}^{d_1})$ and $\mathcal{B}(\mathbf{I}^{d_2})$, respectively. Note that for notational convenience the parameters are not included in the Y's. Let $d = d_1 + d_2$ and let $\{U_i : i \in \mathbf{N}\}$ and $\{V_j : j \in \mathbf{N}\}$ (indicate the location of random points) denote sequences of independent uniformly distributed random variables on $\mathbf{I}^{d_1}$ and $\mathbf{I}^{d_2}$ respectively. The product Poisson process of Y_1 and Y_2 is defined as, for $B \in \mathcal{B}(\mathbf{I}^{d_1+d_2})$,

$$Y_1 \times Y_2(B) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \delta_{(U_i,V_j)}(B),$$

where $N_1 = Y_1(\mathbf{I}^{d_1})$ and $N_2 = Y_2(\mathbf{I}^{d_2})$ (indicate the number of random points) denote Poisson random variables with parameters λ_1 and λ_1 respectively. In section 2, a uniform strong law of large numbers will be proved for a sequence of product Poisson processes.

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Let X and Y be real random variables and let $\{X_{\mathbf{i}} : \mathbf{i} \in \mathbf{N}^{d_1}\}$ and $\{Y_{\mathbf{j}} : \mathbf{j} \in \mathbf{N}^{d_2}\}$ be families of independent identically distributed random variables with $\mathcal{L}(X) = \mathcal{L}(X_{\mathbf{i}})$ and $\mathcal{L}(Y) = \mathcal{L}(Y_{\mathbf{j}})$ respectively. Note that we are not assuming anything between two sequences. Let S_{1n} and S_{2n} be the partial sum processes formed from $\{X_{\mathbf{i}}\}$ and $\{Y_{\mathbf{j}}\}$ and indexed by subsets of $\mathbf{I}^{d_1}$ and $\mathbf{I}^{d_2}$, respectively. Then the product partial sum process corresponding to $\{X_{\mathbf{i}}\}$ and $\{Y_{\mathbf{j}}\}$, indexed by subsets of $\mathbf{I}^d$ with $d = d_1 + d_2$, is defined by

$$S_n(A) := S_n(X, Y, A) := \sum_{|\mathbf{i}| \leq n, |\mathbf{j}| \leq n} X_{\mathbf{i}} Y_{\mathbf{j}} \delta_{(\mathbf{i}/n, \mathbf{j}/n)}(A), \qquad A \subset \mathbf{I}^d.$$

where, $\mathbf{j} = (j_1, j_2, \cdots, j_{d_2})$, $|\mathbf{j}| = max_{1 \leq k \leq d_2} j_k$, $(\mathbf{i}/n, \mathbf{j}/n) = (i_1/n, i_2/n, \cdots, i_{d_1}/n, j_1/n, j_2/n, \cdots, j_{d_2})$ and $\delta_{(\mathbf{i}/n \mathbf{j}/n)}(A) = 1$ or 0 depending on $(\mathbf{i}/n, \mathbf{j}/n) \in A$ or not with i's and j's integers. This product process can be viewed as a special case of dependent partial sum processes, which is much more difficult to deal with than those of independent case. Also this process can be viewed as a generalization of usual partial sum process with $Y_{\mathbf{j}} = 1$ for all $\mathbf{j}$ and $\mathcal{A} = \{B \times \mathbf{I}^{d_2} : B \in \mathcal{B}(\mathbf{I}_{d_2})\}$. For partial sum processes, laws of large number results have been shown to hold; see Bass and Pyke and Giné and Zinn. In section 3 we prove similiar results for a sequence of product partial sum processes S_n under smooth boundary conditions on the index family.

Let $\mathcal{A}$ be a sub-family of $\mathcal{B}(\mathbf{I}^{d_1+d_2})$. Given $A \subset \mathbf{I}^d$, let $A(\delta) = \{x : \rho(x, \partial A)\} < \delta\}$ be the δ-annulus of ∂A, where $\rho(\cdot,\cdot)$ is the Euclidean distance and ∂ denotes the Euclidean boundary of A.

Assumption SBC(Smooth Boundary Condition)

$$r(\delta) := \sup_{A \in \mathcal{A}} |A(\delta)| \to 0 \; as \; \delta \to 0.$$

If, for example, $\mathcal{A}$ were the collection of convex subsets of $\mathbf{I}^d$, it is known to satisfy **SBC**. For this reason this condition is very weak in the sense that: for any d, our theorem will be true, but only for $d = 1, 2$ are the convex subsets a small enough collection for most other purpose, including existence of Brownian processes and uniform convergence results for partial sum processes.

In addition to the strong law of large numbers of this paper, the product partial sum processes also satisfy a uniform central limit theorem and a functional law of iterated logarithm, which will be studied in the forthcoming papers. However, for these later results much stronger conditions, for example moment conditions and metric entropy will be crucial.

2. Product Poisson Processes

Now we state and prove the strong law of large numbers for a sequence of products of Poisson processes under **SBC** on index family.

Theorem 2.1. *Let Y_1 and Y_2 be Poisson processes with integer parameters λ_1 and λ_2 on $\mathcal{B}(\mathbf{I}^{d_1})$ and $\mathcal{B}(\mathbf{I}^{d_2})$ respectively. Assume that $\mathcal{A}$ satisfy Assumption SBC. Then*

$$\text{(I)} \qquad \left\| \frac{Y_1 \times Y_2(\cdot)}{\lambda_1 \lambda_2} - |\cdot| \right\|_{\mathcal{A}} \longrightarrow 0, \ a.s. \quad as \quad \lambda_1, \lambda_2 \longrightarrow \infty,$$

$$\text{(II)} \qquad \left\| \frac{Y_1 \times Y_2(\cdot)}{N_1 N_2} - |\cdot| \right\|_{\mathcal{A}} \longrightarrow 0, \ a.s. \quad as \quad \lambda_1, \lambda_2 \longrightarrow \infty,$$

where $N_1 = Y_1(\mathbf{I}^{d_1})$, $N_2 = Y_2(\mathbf{I}^{d_2})$ and $|\cdot|$ denotes the Lebesgue measure.

Before proving the theorem we introduce some notation following Bass and Pyke (1984). Let m be a fixed positive integer and partition $\mathbf{I}^d$ into regular cubes of side length $1/m$. Let $C_{\mathbf{j}} = \frac{1}{m}(\mathbf{j} - \mathbf{1}, \mathbf{j}]$, where $\mathbf{j} = (j_1, j_2, \cdots, j_d)$ and $\mathbf{1} = (1, 1, \cdots, 1)$ with $1 \le j_k \le m$. Then for any $A \in \mathcal{A}$, define

$$R_m^-(A) = \cup_{C_{\mathbf{j}} \subset A} C_{\mathbf{j}}, \qquad \text{and} \qquad R_m^+(A) = \cup_{C_{\mathbf{j}} \cap A = \emptyset} C_{\mathbf{j}}.$$

That is $R_m^-(A)$ and $R_m^+(A)$ are the inner and the outer rectilinear fits of A by cubes of side length $1/m$. Then since the furthest any point of $R_m^+(A) \setminus R_m^-(A)$ can be from the boundary of A is the diameter of a cube of size $1/m$, by the smooth boundary condition, we have

$$\text{(2.1)} \qquad \sup_{A \in \mathcal{A}} |R_m^+(A) \setminus R_m^-(A)| \le r(d^{1/2}/m)$$

Now define

$$\mathcal{R}_m^- = \{R_m^-(A) | A \in \mathcal{A}\},$$

and

$$\mathcal{R}_m^{\triangle} = \{R_m^+(A) \setminus R_m^-(A) | A \in \mathcal{A}\}.$$

Then, since m is finite, $\sharp(\mathcal{R}_m^-)$ and $\sharp(\mathcal{R}_m^{\triangle})$ are finite respectively. To prove theorem 2.1 we need the following straightforward consequences of known results.

Lemma 2.2. *Let A be rectilinear as defined above, then*

(i) $\dfrac{Y_1(A_{2x})}{\lambda_1} \longrightarrow |A_{2x}|$ *a.s., as* $\lambda_1 \longrightarrow \infty$.

(ii) $\dfrac{Y_1(A_{2x})}{N_1} \longrightarrow |A_{2x}|$ *a.s., as* $\lambda_1 \longrightarrow \infty$.

(iii) $\dfrac{Y_1 \times Y_2(A)}{\lambda_1\lambda_2} \longrightarrow |A|$ *a.s., as* $\lambda_1, \lambda_2 \longrightarrow \infty$.

(iv) $\dfrac{Y_1 \times Y_2(A)}{N_1 N_2} \longrightarrow |A|$ *a.s., as* $\lambda_1, \lambda_2 \longrightarrow \infty$,

where $A_{2x} = \{y \in \mathbf{I}^{d_2} : (x, y) \in A\}$ *is the* d_2*-dimensional section of* A. *And if* $N_1 = 0$ *or* $N_2 = 0$, *then by convention we define* $(Y_1 \times Y_2)/N_1N_2 = 1$.

Proof. The proof is straightforward. For (i), since the set structure is irrelevant, it suffices to show that $X(n)/n \to 1$ a.s. as $n \to \infty$ over the integers, where $X(n)$ is a Poisson random variable with parameter n. And this is a consequence of the Hsu-Robbins SLLN (Hsu and Robbins(1947)) since each $X(n)$ can be expressed as a sum of n independent Poisson random variables with parameter 1, which have a finite second moment.

For (ii),

$$\frac{Y_1(A_{2x})}{N_1} = \frac{Y_1(A_{2x})}{\lambda_1} \cdot \frac{\lambda_1}{N_1}.$$

Since $\lambda_1/N_1 \to 1$ a.s., as $\lambda_1 \to \infty$ (ii) follows from (i).

For (iii) and (iv), it suffices to prove it when A is a rectangle and this is done in (i) and (ii).

Proof of Theorem 2.1. First let $m > 0$ be fixed. Since

$$\begin{aligned}
&\limsup_{\lambda_1,\lambda_2\to\infty, A\in\mathcal{A}} \left|\frac{Y_1 \times Y_2(A)}{\lambda_1\lambda_2} - |A|\right| \\
&\le \limsup_{\lambda_1,\lambda_2\to\infty, A\in\mathcal{A}} \left|\frac{Y_1 \times Y_2(A)}{\lambda_1\lambda_2} - \frac{Y_1 \times Y_2(R_m^-(A))}{\lambda_1\lambda_2}\right| \\
&+ \limsup_{\lambda_1,\lambda_2\to\infty, A\in\mathcal{A}} \left|\frac{Y_1 \times Y_2(R_m^-(A))}{\lambda_1\lambda_2} - |R_m^-(A)|\right| \\
&+ \limsup_{\lambda_1,\lambda_2\to\infty, A\in\mathcal{A}} |A \setminus R_m^-(A)| \\
&:= \mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_3, \qquad (2.2)
\end{aligned}$$

it remains to show that each of T_1, T_2 and $\mathrm{T}_3 \to 0$ as $m \to \infty$. Consider T_1 and T_2

$$T_1 \le \limsup_{\lambda_1,\lambda_2\to\infty, A\in\mathcal{A}} \left|\frac{Y_1 \times Y_2(R_m^+(A) \setminus R_m^-(A))}{\lambda_1\lambda_2}\right|$$

$$
\begin{aligned}
&= \limsup_{\lambda_1,\lambda_2\to\infty, B\in\mathcal{R}_m^{\Delta}} \left| \frac{Y_1 \times Y_2(R_m^+(A) \setminus R_m^-(A))}{\lambda_1\lambda_2} \right| \\
&= \max_{B\in\mathcal{R}_m^{\Delta}} |B| \\
&\leq r(d^{1/2}/m) \qquad \text{a.s.,}
\end{aligned}
$$

the second to last line following from lemma 2.2 (iii). Also by (iii) of lemma 2.2,

$$
\mathrm{T}_2 \leq \limsup_{\lambda_1,\lambda_2\to\infty} \max_{B\in\mathcal{R}_m^-} \left| \frac{Y_1 \times Y_2(B)}{\lambda_1\lambda_2} - |B| \right| = 0 \qquad \text{a.s..}
$$

Finally notice that, by (2.1), $(\mathrm{I}_3) \leq r(d^{1/2}/m)$. Thus by (2.2),

$$
\limsup_{\lambda_1,\lambda_2\to\infty, A\in\mathcal{A}} \left| \frac{Y_1 \times Y_2(A)}{\lambda_1\lambda_2} - |A| \right| \leq 2r(d^{1/2}/m) \text{ a.s.,}
$$

which goes to zero as $m \to \infty$ by Assumption SBC.

For the proof of (II), use (iv) of lemma 2.2 and follow the proof of (I).

Corollary 2.3. *If* $\inf_{A\in\mathcal{A}} |A| > 0$, *then*

(i) $\displaystyle \limsup_{\lambda_1,\lambda_2\to\infty, A\in\mathcal{A}} \left| \frac{Y_1 \times Y_2(A)}{\lambda_1\lambda_2|A|} - 1 \right| = 0 \qquad a.s.$

(ii) $\displaystyle \limsup_{\lambda_1,\lambda_2\to\infty, A\in\mathcal{A}} \left| \frac{Y_1 \times Y_2(A)}{N_1N_2|A|} - 1 \right| = 0 \qquad a.s.$

Proof. This follows from theorem 2.1 by observing

$$
\begin{aligned}
&\limsup_{\lambda_1,\lambda_2\to\infty, A\in\mathcal{A}} \left| \frac{Y_1 \times Y_2(A)}{\lambda_1\lambda_2|A|} - 1 \right| \\
&\leq \limsup_{A\in\mathcal{A}} |A|^{-1} \limsup_{\lambda_1,\lambda_2\to\infty, A\in\mathcal{A}} \left| \frac{Y_1 \times Y_2(A)}{\lambda_1\lambda_2} - |A| \right| = 0 \qquad \text{a.s..}
\end{aligned}
$$

Remark 2.4. In the above we have restricted the parameters to be discrete valued. However the result will also hold for continuous parameters if we impose some further structure. In particular let Y_{iR} be two Poisson processes with parameter 1 defined on $[0,\infty)^{d_i}$, $d_i \geq 1$ and $i = 1,2$. Now suppose that the processes Y_1 and Y_2 in theorem 2.1 are defined, for $A \in \mathcal{B}(\mathbf{I}^{d_1})$ and $B \in \mathcal{B}(\mathbf{I}^{d_2})$, by

$$
Y_1(A) = Y_{1R}({\lambda_1}^{1/d_1} A) \quad \text{and} \quad Y_2(B) = Y_{2R}({\lambda_2}^{1/d_2} B),
$$

then Y_1 and Y_2 are Poisson processes with the right parameters and in this case theorem 2.1 also can be shown to hold.

3. Procuct Partial Sum Processes

In this section we prove a law of large numbers for a sequence of product partial sum processes $\{S_n(X,Y,A) : A \in \mathcal{A}\}$ under conditions on the index family $\mathcal{A}$.

Theorem 3.1. *Let $\{X_{\mathbf{i}} : \mathbf{i} \in \mathbf{N}^{d_1}\}$ and $\{Y_{\mathbf{j}} : \mathbf{j} \in \mathbf{N}^{d_2}\}$ be sequences of independent identically distributed random variables with $EX = \mu_1$, $E|X| < \infty$, $EY = \mu_2$ and $E|Y| < \infty$. Then, under Assumption SBC on $\mathcal{A}$, we have*

$$\left\| n^{-d} S_n(A) - \mu_1\mu_2|A| \right\|_{\mathcal{A}} \longrightarrow 0 \qquad \text{a.s.,} \quad \text{as} \quad n \to \infty.$$

For the proof of theorem 3.1 we prove the following preliminary lemma. Recall the definition of R_m^-, R_m^+, $\mathcal{R}_m^-$ and $\mathcal{R}_m^\triangle$ from section 2.1.

Lemma 3.2. *Let A be a rectilinear subset of $\mathbf{I}^d$. Then, with probability one, as $n \to \infty$,*

$$n^{-d} S_n(A) \longrightarrow \mu_1\mu_2|A|.$$

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ be fixed and write $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$, where $\mathbf{x}_1 = (x_1, x_2, \ldots, x_{d_1})$, $\mathbf{x}_2 = (x_{d_1+1}, x_{d_1+2}, \ldots, x_d)$. i.e. $\mathbf{x}_1 \in \mathbf{I}^{d_1}$, $\mathbf{x}_2 \in \mathbf{I}^{d_2}$. Then

$$(0, \mathbf{x}] = \{(y_1, y_2, y_3, \ldots, y_d) : 0 < y_i \le x_i ; i = 1, 2, 3, \ldots d\},$$

and $|(0,\mathbf{x}]| = |(0,\mathbf{x}_1]| \cdot |(0,\mathbf{x}_2]|$.

Now

$$\frac{S_n((0,\mathbf{x}])}{n^d} = \frac{\sharp(\mathbf{N}^d \cap n(0,\mathbf{x}])}{n^d} \cdot \frac{S_n((0,\mathbf{x}])}{\sharp(\mathbf{N}^d \cap n(0,\mathbf{x}])}.$$

Since

$$S_n((0,\mathbf{x}]) = S_n((0,\mathbf{x}_1] \times (0,\mathbf{x}_2]) = S_{1n}((0,\mathbf{x}_1]) S_{2n}((0,\mathbf{x}_2])$$

and

$$\sharp(\mathbf{N}^d \cap n(0,\mathbf{x}]) = \sharp(\mathbf{N}^{d_1} \cap n(0,\mathbf{x}_1]) \cdot \sharp(\mathbf{N}^{d_2} \cap n(0,\mathbf{x}_2]),$$

we have, by the classical strong law of large numbers,

$$\begin{aligned}\frac{S_n((0.\mathbf{x}])}{n^d} &= \frac{\sharp(\mathbf{N}^d \cap n(0,\mathbf{x}])}{n^d} \cdot \frac{S_{1n}(n(0,\mathbf{x}_1])S_{2n}(n(0,\mathbf{x}_2])}{\sharp(\mathbf{N}^{d_1} \cap n(0,\mathbf{x}_1]) \cdot \sharp(\mathbf{N}^{d_2} \cap n(0,\mathbf{x}_2])} \\ &\longrightarrow |(0,\mathbf{x}]|\mu_1\mu_2 \qquad \text{a.s.,}\end{aligned}$$

as $n \to \infty$.

But, since any rectilinear set can be obtained by a finite number of unions and differences of rectangles of the form $(0,\mathbf{x}]$, by linearity we have

$$n^{-d}S_n(A) \longrightarrow \mu_1\mu_2|A| \quad a.s.,$$

as $n \to \infty$.

Proof of Theorem 3.1. The proof is quite similiar to the case of product of Poisson processes.

$$\begin{aligned}&\limsup_{n\to\infty, A\in\mathcal{A}} \left|n^{-d}S_n(A) - \mu_1\mu_2|A|\right| \\ &\quad= \limsup_{n\to\infty, A\in\mathcal{A}} n^{-d}\left|S_n(A) - S_n(R_m^-(A))\right| \\ &\qquad+ \limsup_{n\to\infty, A\in\mathcal{A}} \left|n^{-d}S_n(R_m^-(A)) - \mu_1\mu_2|R_m^-(A)|\right| \\ &\qquad+ \limsup_{n\to\infty, A\in\mathcal{A}} \mu_1\mu_2|A \setminus R_m^-(A)| \\ &\quad= \mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_3.\end{aligned}$$

Clearly, $\mathrm{T}_3 \le \mu_1\mu_2 r(d^{1/2}/m)$ since $\sharp\mathcal{R}_m^- < \infty$. Also

$$\begin{aligned}\mathrm{T}_2 &\le \limsup_{n\to\infty, B\in\mathcal{R}_m^-} \left|n^{-d}S_n(B) - \mu_1\mu_2|B|\right| \\ &\le \limsup_{n\to\infty} \max_{B\in\mathcal{R}_m^-} \left|n^{-d}S(B) - \mu_1\mu_2|B|\right| = 0 \text{ a.s.}\end{aligned}$$

Finally, let $\alpha = E|X|$ and $\beta = E|Y|$. For $C \subset \mathbf{I}^d$, set

$$T_n(C) = \sum_{|\mathbf{i}|<n, |\mathbf{j}|<n} |X_\mathbf{i}||Y_\mathbf{j}|\delta_{(\mathbf{i}/n, \mathbf{j}/n)}(C).$$

By lemma 3.2 applied to the process T_n.

$$\begin{aligned}\mathrm{T}_1 &\le \limsup_{n\to\infty, A\in\mathcal{A}} n^{-d}T_n(R_m^+(A) \setminus R_m^-(A)) \\ &\le \limsup_{n\to\infty} \max_{B\in\mathcal{R}_m^\Delta} \left|n^{-d}T_n(B)\right| \\ &\le \alpha\beta \max_{B\in\mathcal{R}_m^\Delta} |B| \le \alpha\beta r(d^{1/2}/m) \qquad \text{a.s..}\end{aligned}$$

Summing up and letting $m \to \infty$, we have the conclusion.

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